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ON THE ZILBER-PINK CONJECTURE FOR COMPLEX ABELIAN VARIETIES

BY Fabrizio BARROERO AND Gabriel A. DILL

ABSTRACT. – In this article, we prove that the Zilber-Pink conjecture for abelian varieties over an arbitrary field of characteristic 0 is implied by the same statement for abelian varieties over the field of algebraic numbers.

More precisely, the conjecture holds for subvarieties of dimension at most m in the abelian variety A if it holds for subvarieties of dimension at most m in the largest abelian subvariety of A that is isomorphic to an abelian variety defined over $\bar{\mathbb{Q}}$.

RÉSUMÉ. – Dans cet article, nous prouvons que la conjecture de Zilber-Pink pour les variétés abéliennes sur un corps quelconque de caractéristique 0 est impliquée par le même énoncé pour les variétés abéliennes sur le corps des nombres algébriques.

Plus précisément, la conjecture est vraie pour les sous-variétés de dimension inférieure ou égale à m dans la variété abélienne A si elle est vraie pour les sous-variétés de dimension inférieure ou égale à m dans la plus grande sous-variété abélienne de A qui est isomorphe à une variété abélienne définie sur $\bar{\mathbb{Q}}$.

1. Introduction

For us, varieties and curves are irreducible and subvarieties are always irreducible and closed in the ambient variety. Fields are always of characteristic 0. We work with the Zariski topology, therefore by open, dense, etc. we always mean Zariski open, Zariski dense, etc. except when we consider connected mixed Shimura (sub)varieties or (sub)data.

Let A be an abelian variety defined over an algebraically closed field K. A special subvariety of A is an irreducible component of an algebraic subgroup of A or, equivalently, a translate of an abelian subvariety by a torsion point. Arbitrary translates of abelian subvarieties are called cosets or weakly special subvarieties. Special subvarieties are also called torsion cosets.

The Manin-Mumford conjecture, proven by Raynaud [32], states that a subvariety of an abelian variety contains at most finitely many maximal special subvarieties. In particular, a non-special curve contains at most finitely many torsion points.

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Given a curve in an abelian variety, a dimension count suggests that it should not intersect a special subvariety of codimension at least 2. If one considers the union of all special subvarieties of codimension at least 2 and intersects it with a curve that is not contained in a proper special subvariety, one expects the intersection to be finite.

The pioneering work [3] of Bombieri, Masser and Zannier was one of the first to study this kind of problems and to pass from considering torsion points in subvarieties of algebraic groups to points lying in algebraic subgroups of appropriate codimension.

Indeed, Bombieri, Masser and Zannier proved that, given a curve defined over the algebraic numbers and contained in $\mathbb{G}_m^n$ but not in any of its proper (not necessarily torsion) cosets, it contains at most finitely many points that lie in an algebraic subgroup of $\mathbb{G}_m^n$ of codimension at least 2. The condition of not being contained in a proper coset was replaced by the necessary one of not lying in a proper torsion coset by Maurin [24] and independently by Bombieri, Habegger, Masser and Zannier in [2].

In the same paper [3], Bombieri, Masser and Zannier suggest that a possible analogue of their result for curves in $\mathbb{G}_m^n$ could hold for (families of) abelian varieties and that one could consider higher dimensional subvarieties and intersect them with algebraic subgroups of higher codimension.

A couple of years later, Zilber [43] independently stated a conjecture for semiabelian varieties of which the result of Bombieri, Masser and Zannier is a consequence. This is formulated in slightly different language and we are going to state it later. Similar conjectures for $\mathbb{G}_m^n$ were formulated by Bombieri, Masser and Zannier in [4].

We now consider an apparently weaker formulation of the same principle due to Pink. We introduce the following notation: For a non-negative integer k, we denote by $A^{[k]}$ the union of all special subvarieties of A of codimension at least k.

Pink conjectured in [28] that, if $V \cap A^{[\dim V+1]}$ is Zariski dense in V for a subvariety V of A, then V is contained in a proper special subvariety of A. The conjecture in its full generality is still open. If V is a curve and $K = \bar{\mathbb{Q}}$, it has been proven by Habegger and Pila in [21]. Previously, partial results have been obtained by Viada [40], [41], Rémond and Viada [36], Ratazzi [31], Carrizosa [6], [7] in combination with Rémond [33], [34], [35], and Galateau [12]. If V is a hypersurface, Pink's conjecture follows from the Manin-Mumford conjecture. If the dimension and codimension of V are at least 2, then all known results place additional restrictions on V or A, see for instance [8], [9], and [23].

In this article, we use a recent result of Gao in [15], which generalizes work by Rémond in [35], to reduce Zilber's conjecture to the case where everything is defined over $\bar{\mathbb{Q}}$. We even show that it can be reduced to Pink's formulation of the conjecture over $\bar{\mathbb{Q}}$. Furthermore, we prove the full conjecture in Corollary 1.7 if no abelian variety of dimension greater than 4 that is defined over $\bar{\mathbb{Q}}$ embeds into A. For example, the conjecture holds in a power of an elliptic curve with transcendental j-invariant. Combining Theorem 1.5 below with Theorem 1.1 of [21] yields the following theorem:

THEOREM 1.1. – *Let A be an abelian variety defined over an algebraically closed field K (of characteristic 0) and let $V \subset A$ be a curve. Then $V \cap A^{[2]}$ is finite unless V is contained in a proper algebraic subgroup of A.*

As mentioned before, Pink's conjecture is implied by the following Conjecture 1.2 on unlikely or atypical intersections that was formulated by Zilber in [43] for semiabelian varieties. An overview of the topic of unlikely intersections is given in the book [42].

In order to state Conjecture 1.2, we introduce the notion of an atypical subvariety: Let A be an abelian variety defined over an algebraically closed field K and let V be a subvariety of A. A subvariety W of V is called atypical (for V in A) if W is an irreducible component of the intersection of V with a special subvariety of codimension at least $\dim V - \dim W + 1$. It is called maximal if it is not contained in any larger atypical subvariety.

Conjecture 1.2. – *Let K be an algebraically closed field. Let A be an abelian variety defined over K and let V be a subvariety of A. Then V contains at most finitely many maximal atypical subvarieties.*

If V is a curve, then Conjecture 1.2 and Pink's conjecture are obviously equivalent.

It turns out that another equivalent formulation of Conjecture 1.2 is more suited to our proof strategy. In order to state it, we have to introduce the notions of defect and optimality of a subvariety.

Definition 1.3. – *If V is a subvariety of A, then there is a smallest special subvariety $\langle V\rangle$ containing V. We define the defect $\delta(V)$ of V to be $\dim\langle V\rangle - \dim V$. A subvariety W of V is called optimal for V in A if $\delta(U) > \delta(W)$ for every subvariety U with $W \subsetneq U \subset V$.*

Pink introduced the notion of defect in [28], while the concept of optimality was introduced in [21] by Habegger and Pila. The latter is motivated by Poizat's notion of cd-maximality in [29]. cd-maximality is the toric analogue of the notion of geodesic optimality, which we will introduce later. Using the concept of optimality, Habegger and Pila formulated the following conjecture, which is equivalent to Conjecture 1.2 by Lemma 2.7 of [21].

Conjecture 1.4. – *Let K be an algebraically closed field and let d be a non-negative integer. Let A be an abelian variety defined over K and let V be a subvariety of A. Then V contains at most finitely many optimal subvarieties of defect at most d.*

In the statement of our results, we use the trace of an abelian variety with respect to a field extension of algebraically closed fields. This can be thought of as the largest abelian subvariety defined over the smaller field. See Definition 2.3 for a formal definition.

The following is the main result of this article:

Theorem 1.5. – *Let K be an algebraically closed field, let m be a non-negative integer and A an abelian variety defined over K with $K/\bar{\mathbb{Q}}$-trace (T, Tr). Then, if Conjecture 1.4 holds for some non-negative integer d and subvarieties of dimension at most m in T (over the field $\bar{\mathbb{Q}}$), it holds for the same d and subvarieties of dimension at most m in A (over K).*

Note that Habegger and Pila have shown in Corollary 9.10 of [21] that Conjecture 1.4 can be further reduced to the existence of sufficiently strong lower bounds for the size of the Galois orbits of optimal singletons over a field of definition that is finitely generated over $\mathbb{Q}$.

An analogue of Theorem 1.5 for powers of the multiplicative group was proven in [5] by Bombieri, Masser and Zannier. Note that in this case the ambient algebraic group is always