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GLOBAL BEHAVIORS OF DEFOCUSING SEMILINEAR WAVE EQUATIONS

BY SHIWU YANG

ABSTRACT. – In this paper, we investigate the global behaviors of solutions to defocusing semilinear wave equations in $\mathbb{R}^{1+d}$ with $d \geq 3$. We prove that in the energy space the solution verifies the integrated local energy decay estimates for the full range of energy subcritical and critical powers. For the case where $p > \frac{d+1}{d-1}$, we derive a uniform weighted energy bound for the solution as well as inverse polynomial decay of the energy flux through hypersurfaces away from the light cone. As a consequence, the solution scatters in the energy space and in the critical Sobolev space for p with an improved lower bound. This in particular extends the existing scattering results to higher dimensions without spherical symmetry.

RÉSUMÉ. – Dans cet article, nous étudions les comportements globaux des solutions aux équations des ondes semi-linéaires défocalisantes dans $\mathbb{R}^{1+d}$ avec $d \geq 3$. Nous prouvons que dans l'espace d'énergie la solution vérifie les estimations intégrées de la décroissance de l'énergie locale pour l'ensemble des cas énergie sous-critiques et critiques. Pour le cas où $p > \frac{d+1}{d-1}$, nous dérivons une borne d'énergie pondérée uniforme pour la solution ainsi que la décroissance polynomiale inverse du flux d'énergie à travers des hypersurfaces en dehors du cône de lumière. En conséquence, la solution se disperse dans l'espace d'énergie et dans l'espace de Sobolev critique pour p avec une borne inférieure améliorée. Cela étend en particulier les résultats de diffusion existant à des dimensions supérieures sans symétrie sphérique.

1. Introduction

In this paper, we study the global asymptotic behaviors for solutions of the following defocusing semilinear wave equation

$$\Box\phi = |\phi|^{p-1}\phi, \quad \phi(0,x) = \phi_0(x), \quad \partial_t\phi(0,x) = \phi_1(x) \tag{1}$$

with energy subcritical or energy critical power $1 < p \leq \frac{d+2}{d-2}$ in $\mathbb{R}^{1+d}$.

This simple nonlinear model has drawn extensive attention in the past decades. Existence of global classical C^2 solutions had been obtained early in [24] with energy subcritical smooth nonlinearity in dimension $d = 3$. Extensions and generalizations could be found for

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example in [9], [10], [36], [39], [49], [44]. These results aimed at showing the local boundedness of the solution with sufficiently regular initial data. However since the power p is close to 1 in higher dimensions, the above mentioned results only hold in lower dimensions $d \leq 9$. This calls for a global well-posedness result in the more natural energy space, which was addressed by Ginibre-Velo in [16], [18] for the full energy subcritical case in all dimensions. This type of result is indeed a local existence result due to the conservation of energy and nothing too much could be said on the global and asymptotic behaviors of the solutions.

For the energy critical case, in dimension $d = 3$, Struwe in [45] showed that the solution exists globally in time if the data are spherically symmetric. Later Grillakis in [20] removed this symmetry assumption and obtained the global regularity of the solution, that is, the solution is smooth if the initial data are. He also derived asymptotic pointwise decay estimates for the solution by using conformal transformations. This global regularity result was extended to higher dimensions up to $d \leq 9$ in [21], [40], [27].

On the other hand, in the energy space, Kapitanski [25] showed the existence and uniqueness of global weak solutions in the energy space and obtained partial regularity for the solution in [26]. The global well-posedness in energy space was finally accomplished by Shatah-Struwe in [41]. A key observation leading to these global existence results is the non-concentration of energy. Bahouri-Shatah [6] could even show that the potential part of energy decays to zero, which was applied to prove that the solution scatters to free linear wave by Bahouri-Gérard in [5].

Due to the lack of rigidity compared to the energy critical case, asymptotic properties for the solution in the energy subcritical case usually require additional restrictions on the initial data and the lower bound of the power p. In dimension $d = 3$ with sufficiently smooth and localized initial data, pointwise time decay of the solution has been derived in [44], [48], [7], [8] for the superconformal case $p \geq \frac{d+3}{d-1}$. Weaker decay estimates were achieved in [38], [19] for part of the subconformal case. These pointwise decay estimates in lower dimensions mainly relied on the approximate conservation of conformal energy (arising from the conformal symmetry of Minkowski space) as well as the representation formula of linear wave equations. However, such a method fails in higher dimensions as the conserved energy is too weak to control the nonlinearity. Alternatively there have been plenty of literatures on the scattering theory of the solution, aiming at comparing the solutions of nonlinear equations to those of linear equations at time infinity. A complete scattering theory consists of constructing a wave operator and proving the asymptotic completeness, that is, the solution behaves like a linear solution at time infinity in a suitable function space (see detailed discussions in [17]). For the problems we are concerned with, there are mainly two issues we need to address in order to establish such a theory. The first one is to find a correct function space in which the nonlinear solutions and the associated linear solutions are compared. The other one is to seek the lowest possible power p so that the scattering theory still holds. This is closely tied to the decay properties of linear waves in higher dimensions, that is, larger power leads to faster decay of the nonlinearity. Based on the approximate conservation of conformal energy, Ginibre-Velo [17] obtained time decay for the solutions in the conformal energy space (weighted energy space with weights $(1 + |x|^2)$) for the superconformal case, also see an alternative treatment in [4]. In lower dimensions $2 \leq d \leq 4$, they also covered part of the subconformal case. These uniform time decay properties are sufficiently strong to establish the complete scattering theory, which was later extended to higher dimensions $d \leq 6$ by Hidano [23], [22]. However, it is not clear whether the lower bound given in these works is sharp or applies to even higher dimensions since the lower bound of p for the asymptotic completeness is larger than that for existence of wave operator (see for example [23]).

One reason that p is more restrictive for asymptotic completeness is that the norm of the chosen function space (H^1 with finite conformal energy) is too strong. The lower bound on p could indeed be greatly improved for asymptotic completeness if the function space is enlarged to be the energy space $\dot{H}^1$, see [19], [37], [38]. However these results still required that the initial data belong to the conformal energy space.

Another important intermediate function space to study the asymptotic completeness is the critical Sobolev space $\dot{H}^{s_p}$, which is partially motivated by the open problem that is whether the solution to the nonlinear Equation (1) exists globally and scatters in $\dot{H}^{s_p}$. Dodson [13], [14] first gave an affirmative answer to this problem for the superconformal case $3 \leq p < 5$ in dimension $d = 3$ under spherical symmetry. A conditional result, that is, the uniform boundedness of the critical Sobolev norm of the solution implies global existence and scattering, has been established in [15] without spherical symmetry. For data in some weighted energy space which belongs to the critical Sobolev space but contains the conformal energy space, Shen in [42] proved that the solution scatters in $\dot{H}^{s_p}$. However this result still requires spherical symmetry and is only for the superconformal case in dimension $d = 3$.

The above mentioned results regarding the asymptotic decay properties of the solutions do not hold for the one dimensional case $d = 1$, $p > 1$, as shown by Lindblad-Tao [32] that the solution exhibits a type of weak averaged decay estimate, which is clearly not shared by linear waves. In particular, for $d = 1$, the solution does not approach linear ones as higher dimensional cases.

The aim of the present paper is to find new evidences that solutions to the energy subcritical and critical defocusing nonlinear wave equations behave like linear waves for $d \geq 3$ with initial data in some weighted energy space larger than the conformal energy space. There are several types of estimates that can characterize the global behavior of linear waves, of which the weakest version is the integrated local energy decay estimates. This type of estimate gives a uniform spacetime bound for the solution in terms of the initial energy and recently has been widely used to study linear waves on general Lorentzian manifolds, including black hole spacetimes, see for example [11], [43], [47]. We show that for the full range of energy subcritical and critical case $1 < p \leq \frac{d+2}{d-2}$ solutions to (1) verify the integrated local energy decay estimates, which in particular implies that the energy can not concentrate at a point. This fact is crucial to conclude the global well-posedness result for the energy critical equations.

Since linear waves travel along outgoing light cones, the energy flux through hypersurfaces away from the light cone decays in terms of the distance to the light cone as shown in [12]. We demonstrate that for the case where $\frac{d+1}{d-1} < p \leq \frac{d+2}{d-2}$ quantitative energy flux decay estimates hold for solutions of (1). As a consequence, we have the uniform spacetime bound for the potential $|\phi|^{p+1}$, which leads to the scattering result in critical Sobolev space and in energy space with improved lower bound on p than those in [17], [23], [22] mentioned above. Moreover, our result applies to all higher dimensions $d \geq 3$ without spherical symmetry assumption, hence refining the scattering result of Shen in [42].