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Philippe BOUGEROL & Manon DEFOSSEUX

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Annales Scientifiques de l'École Normale Supérieure,
45, rue d'Ulm, 75230 Paris Cedex 05, France.
Tél. : (33) 1 44 32 20 88. Fax : (33) 1 44 32 20 80.
Email : `annales@ens.fr`

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Société Mathématique de France
Case 916 - Luminy
13288 Marseille Cedex 09
Tél. : (33) 04 91 26 74 64. Fax : (33) 04 91 41 17 51
Email : `abonnements@smf.emath.fr`

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PITMAN TRANSFORMS AND BROWNIAN MOTION IN THE INTERVAL VIEWED AS AN AFFINE ALCOVE

BY Philippe BOUGEROL AND Manon DEFOSSEUX

ABSTRACT. – Pitman's theorem states that if $\{B_t, t \geq 0\}$ is a one dimensional Brownian motion, then $\{B_t - 2\inf_{0\leq s\leq t} B_s, t \geq 0\}$ is a three dimensional Bessel process, i.e., a Brownian motion conditioned to remain forever positive. This paper gives a similar representation for the Brownian motion conditioned to remain in a given interval. Due to the double barrier condition, this representation is more involved and only asymptotic. One uses the fact that the interval is an alcove of the Kac-Moody affine Lie algebra $A_1^{(1)}$, the Littelmann path approach of representation theory and a dihedral approximation.

RÉSUMÉ. – Le théorème de Pitman affirme que si $\{B_t, t \geq 0\}$ est un mouvement brownien unidimensionnel, alors $\{B_t - 2\inf_{0\leq s\leq t} B_s, t \geq 0\}$ est un processus de Bessel de dimension trois, c'est-à-dire un brownien conditionné à rester positif. Nous donnons dans cet article une représentation analogue pour le brownien conditionné à rester dans un intervalle donné. En raison de la présence de deux extrémités, cette représentation est plus compliquée que celle du théorème original. Nous utilisons le fait que l'intervalle est une alcôve pour l'algèbre de Kac-Moody affine $A_1^{(1)}$, l'approche par le modèle de chemins de Littelmann de la théorie des représentations et une approximation diédrale.

1. Introduction

1.1. – The probability transition of the Brownian motion conditioned to stay positive forever is the Doob transform of the difference of two heat kernels. This is a consequence of the reflection principle at 0. Pitman's theorem [31] of 1975 gives the path representation of this process as

$$\mathcal{P}B(t) = B_t - 2\inf_{0\leq s\leq t} B_s,$$

where B is a standard Brownian motion with $B_0 = 0$. The transform $\mathcal{P}B$ is written with the reflection at 0. Consider now a Brownian motion conditioned to stay in the interval [0, 1] forever. Is it possible to write it as a path transform of a Brownian motion B by some kind of folding? The conditioned process can be seen as the Doob transform of the Brownian motion killed at 0 and at 1. Its probability transition is an alternating infinite sum which can

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be obtained by applying successive reflection principles at 0 and at 1 (method of images). It is therefore natural to ask if Pitman's theorem has an analogue for the conditioned process in the interval written with two similar transforms at 0 and 1, maybe repeated an infinite number of times. The main result of this article is to show that, to our surprise, this is not exactly the case. A small correction (a Lévy transform) has to be added: this is due to the non differentiabilty of the Brownian motion. Interestingly, the same correction also occurs in an asymptotic property of the highest weight representations of the affine Lie algebra $A_1^{(1)}$. Hence there are deep links between the trajectories of the Brownian motion and representation theory of Kac-Moody algebras.

1.2. – Let us state our main result. We suppose that $\mu \in [0,1]$. We will give in Section 2.1 a precise definition of the following process.

NOTATION 1.1. – *Z^μ is a Brownian motion conditioned to stay in $[0,1]$ forever such that $Z_0^\mu = \mu$.*

We consider, for a continuous real path $\varphi : \mathbb{R}_+ \to \mathbb{R}$ such that $\varphi(0) = 0$, for $t \geq 0$,

$$\mathcal{L}_1\varphi(t) = \varphi(t) - \inf_{0\leq s\leq t} \varphi(s),$$
$$\mathcal{P}_1\varphi(t) = \varphi(t) - 2\inf_{0\leq s\leq t} \varphi(s).$$

We call them the classical Lévy and Pitman transforms of φ. We introduce

$$\mathcal{L}_0\varphi(t) = \varphi(t) + \inf_{0\leq s\leq t}(s - \varphi(s)),$$
$$\mathcal{P}_0\varphi(t) = \varphi(t) + 2\inf_{0\leq s\leq t}(s - \varphi(s)).$$

For $n \in \mathbb{N}$, we let $\mathcal{P}_{2n} = \mathcal{P}_0, \mathcal{L}_{2n} = \mathcal{L}_0$ and $\mathcal{P}_{2n+1} = \mathcal{P}_1, \mathcal{L}_{2n+1} = \mathcal{L}_1$. The aim of this paper is the following representation theorem (see Theorem 7.5).

THEOREM. – *Let $\mu \in [0,1]$ and let $B_t^\mu = B_t + t\mu$ be the real Brownian motion with drift μ starting from 0. For any $t > 0$, almost surely,*

$$\lim_{n\to\infty} t\mathcal{L}_{n+1}\mathcal{P}_n\cdots\mathcal{P}_1\mathcal{P}_0 B^\mu(1/t) = \lim_{n\to\infty} t\mathcal{L}_{n+1}\mathcal{P}_n\cdots\mathcal{P}_2\mathcal{P}_1 B^\mu(1/t) = Z_t^\mu.$$

1.3. – Briefly, the strategy of the proof is as follows. The first step is to linearize the problem (the reflection at 1 is not linear): we introduce the process $A^{(\mu)}$ which is the space-time Brownian motion $B_t^{(\mu)} = (t, B_t^\mu), t \geq 0$, conditioned to stay in the affine cone

$$C_{\text{aff}} = \{(t,x) \in \mathbb{R}^2; 0 < x < t\}.$$

We show that Z^μ is the space component of the time inverted process of $A^{(\mu)}$, i.e., $A_t^{(\mu)} = (t, tZ_{1/t}^\mu)$ in distribution. So we will work essentially with $A^{(\mu)}$.

We define a sequence of non-negative random processes $\xi_n(t), t \geq 0, n \in \mathbb{N}$, by

$$\xi_n(t) = -\inf_{0\leq s\leq t}\{s1_{2\mathbb{N}}(n) + (-1)^{n-1}\mathcal{P}_{n-1}\cdots\mathcal{P}_0 B^\mu(s)\}. \tag{1.1}$$

Then

$$\mathcal{P}_n\cdots\mathcal{P}_0 B^\mu(t) = B_t^\mu + 2\sum_{k=0}^{n}(-1)^{k+1}\xi_k(t), \tag{1.2}$$

and

$$\mathcal{L}_{n+1}\mathcal{P}_n \cdots \mathcal{P}_0 B^{\mu}(t) = \mathcal{P}_n \cdots \mathcal{P}_0 B^{\mu}(t) + (-1)^n \xi_{n+1}(t). \tag{1.3}$$

We first suppose that $\mu \neq 0, 1$. Then the random variables

$$\xi_n(\infty) = \lim_{t \to +\infty} \xi_n(t)$$

are finite a.s. and their distributions have a simple explicit representation with independent exponential random variables. The properties of $\xi_n(t), n \in \mathbb{N}$, can be deduced by conditioning arguments. This allows us to show that for all $t \geq 0$ the limit of (1.3) exists a.s. and has the distribution of the space component of $A^{(\mu)}$. We also prove that for $t > 0$, $\xi_n(t)$ tends to 2 almost surely when n tends to $+\infty$. This shows that the limit of (1.2) itself does not exist. The boundary cases $\mu = 0, 1$, are dealt with using the Cameron–Martin–Girsanov (CMG) theorem.

To prove these results, we approximate the space-time Brownian motion $B^{(\mu)}$ by planar Brownian motions with proper drifts and we approximate $A^{(\mu)}$ by these planar Brownian motions conditioned to remain in a wedge in $\mathbb{R}^2$ of dihedral angle π/m. The dihedral case has been dealt with in Biane et al. [2] and we use their results. Due to the need of the correction term, the approximation is not immediate.

1.4. – The article is organized as follows. We always suppose that $0 \leq \mu \leq 1$. In Section 2 we first define rigorously Z^{μ}. Then we define $A^{(\mu)}$ the conditioned space-time Brownian motion with drift μ in the affine Weyl cone C_{aff}. We prove in Theorem 2.4 that, as processes, $A_t^{(\mu)} = (t, tZ^{\mu}_{1/t})$ in distribution. In Section 3 we recall the Pitman representation theorem for planar Brownian motions in a dihedral cone and show how they approximate $A^{(\mu)}$. In Section 4 we introduce the string parameters in the dihedral case. The analogous parameters $\{\xi_n(t), n \in \mathbb{N}, t \geq 0\}$ for the space time Brownian motion $B^{(\mu)}$ are defined in Section 5 and called the affine string parameters. When $\mu \neq 0, 1$, then $\xi_n(\infty), n \in \mathbb{N}$, are finite and are called the Verma affine parameters. In Section 6 we study the highest weight process $\Lambda^{(\mu)}$ which is the limit of the image of $B^{(\mu)}$ under the transformation (1.3). It is shown in Section 7 that $\Lambda^{(\mu)}$ equals $A^{(\mu)}$ in distribution. The representation theorem for the Brownian motion Z^{μ} in [0, 1] follows. In Section 8 we first compute the conditional distribution of $B_t^{(\mu)}$ given the sigma-algebra $\sigma\{\Lambda^{(\mu)}(s), s \leq t\}$ and then the distributions of $L^{(\mu)}(\infty)$ and $\xi_1(\infty)$. Up to this point we only use probabilistic arguments with no reference to algebra.

In Section 9, we introduce the infinite dimensional affine Lie algebra $A_1^{(1)}$ and show how our results are related to its highest weight representations. We show that the conditional distribution of the Brownian motion is a Duistermaat Heckman measure for a circle action. It describes the semiclassical behavior of the weights of a representation when its highest weight is large. The Lévy correction term also occurs in the behavior of the elements of large weight of the Kashiwara crystal $B(\infty)$, which is of independent interest.

1.5. – We have chosen to present the proof of our probabilistic results without explicit reference to Kac-Moody algebra, so that it can be read by probabilists. But let us now explain the ideas of representation theory behind the scenes because this has been a source of inspiration. This may be helpful for some readers. This also explains our choice of terminolgy (affine cone, string parameter, highest weight,…).