ISSN 0012-9593

quatrième série - tome 55 fascicule 2 mars-avril 2022

ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

Emanuele DOTTO, Irakli PATCHKORIA & Kristian JONSSON MOI

Witt Vectors, Polynomial Maps, and Real Topological Hochschild Homology

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Annales Scientifiques de l'École Normale Supérieure

Publiées avec le concours du Centre National de la Recherche Scientifique

Responsable du comité de rédaction / *Editor-in-chief*

Yves de Cornulier

Publication fondée en 1864 par Louis Pasteur

Continuée de 1872 à 1882 par H. Sainte-Claire Deville
de 1883 à 1888 par H. Debray
de 1889 à 1900 par C. Hermite
de 1901 à 1917 par G. Darboux
de 1918 à 1941 par É. Picard
de 1942 à 1967 par P. Montel

Comité de rédaction au 1er octobre 2021

S. Cantat
G. Carron
Y. Cornulier
F. Déglise
A. Ducros
B. Fayad
G. Giacomin
D. Häfner
D. Harari
C. Imbert
S. Morel
P. Shan

Rédaction / *Editor*

Annales Scientifiques de l'École Normale Supérieure,
45, rue d'Ulm, 75230 Paris Cedex 05, France.
Tél. : (33) 1 44 32 20 88. Fax : (33) 1 44 32 20 80.
Email : `annales@ens.fr`

Édition et abonnements / *Publication and subscriptions*

Société Mathématique de France
Case 916 - Luminy
13288 Marseille Cedex 09
Tél. : (33) 04 91 26 74 64. Fax : (33) 04 91 41 17 51
Email : `abonnements@smf.emath.fr`

Tarifs

Abonnement électronique : 441 euros.
Abonnement avec supplément papier :
Europe : 619 €. Hors Europe : 698 € ($ 985). Vente au numéro : 77 €.

© 2022 Société Mathématique de France, Paris

En application de la loi du 1er juillet 1992, il est interdit de reproduire, même partiellement, la présente publication sans l'autorisation de l'éditeur ou du Centre français d'exploitation du droit de copie (20, rue des Grands-Augustins, 75006 Paris).
All rights reserved. No part of this publication may be translated, reproduced, stored in a retrieval system or transmitted in any form or by any other means, electronic, mechanical, photocopying, recording or otherwise, without prior permission of the publisher.

ISSN 0012-9593 (print) 1873-2151 (electronic)

Directeur de la publication : Fabien Durand
Périodicité : 6 nos / an

Ann. Scient. Éc. Norm. Sup.
4^{e} série, t. 55, 2022, p. 473 à 535

WITT VECTORS, POLYNOMIAL MAPS, AND REAL TOPOLOGICAL HOCHSCHILD HOMOLOGY

BY Emanuele DOTTO, Irakli PATCHKORIA
AND Kristian JONSSON MOI

Abstract. – We show that various flavours of Witt vectors are functorial with respect to multiplicative polynomial laws of finite degree. We then deduce that the p-typical Witt vectors are functorial in multiplicative polynomial *maps* of degree at most $p-1$. This extra functoriality allows us to extend the p-typical Witt vectors functor from commutative rings to $\mathbb{Z}/2$-Tambara functors, for odd primes p. We use these Witt vectors for Tambara functors to describe the components of the dihedral fixed-points of the real topological Hochschild homology spectrum at odd primes.

Résumé. – On prouve que différents types de vecteurs de Witt sont fonctoriels en lois polynôme de degré fini. On en déduit que les vecteurs de Witt p-typiques sont fonctoriels en applications polynôme de degré au plus $p-1$. Cette fonctorialité nous permet d'étendre les vecteurs de Witt p-typiques des anneaux commutatifs aux foncteurs de Tambara pour le group $\mathbb{Z}/2$, quand p est un nombre premier impair. On utilise ces vecteurs de Witt pour décrire les composantes des points-fixes diédraux de l'homologie de Hochschild topologique réelle aux premiers impairs.

Introduction

The various rings of Witt vectors have a prominent role in number theory and algebraic topology. They are defined as endofunctors on the category of commutative rings, and they provide a functorial way of passing from characteristic p to characteristic zero. The prototypical example is the ring of p-typical Witt vectors of the field $\mathbb{F}_p$, which is isomorphic to the ring of p-adic integers $\mathbb{Z}_p$. These Witt vectors functors exhibit some fundamental extra structures, such as λ-operations, δ-ring structures, and Frobenius lifts, which determine several of their universal properties (see e.g., [4], [22]). In topology, Witt vectors appear in calculations related to topological cyclic homology [20], cyclic K-theory [1], and in chromatic homotopy theory. Here they also exhibit extra structure, as they relate to the free Tambara functors of the cyclic groups [9]. In this paper we will provide novel additional structure on the Witt vectors, related to polynomial laws and polynomial maps.

0012-9593/02/© 2022 Société Mathématique de France. Tous droits réservés doi:10.24033/asens.2500

We recall from [33] that a *multiplicative polynomial law* f from a commutative ring A to a commutative ring B is a collection of multiplicative maps

$$f_R \colon A \otimes_{\mathbb{Z}} R \longrightarrow B \otimes_{\mathbb{Z}} R$$

for every commutative ring R, which is natural with respect to ring homomorphisms in R. Every multiplicative polynomial law f of finite degree n has an underlying multiplicative map

$$f_{\mathbb{Z}} \colon A \longrightarrow B,$$

which is n-polynomial, in the sense that its $(n+1)$-st cross-effect, or deviation, vanishes. The main goal of this paper is to show that various Witt vectors functors extend from the category of commutative rings and ring homomorphisms to the category of commutative rings and polynomial laws of finite degree, or to polynomial maps.

In §1 we introduce an axiomatic framework of "PD-functors," to study this extended functoriality in polynomial laws. A *PD-functor* is an endofunctor of the category of commutative rings

$$F \colon \mathrm{Ring} \longrightarrow \mathrm{Ring},$$

which commutes with certain limits and colimits. Examples of these functors include the Witt vectors W_S for any truncation set $S \subset \mathbb{N}$, so in particular the big and p-typical Witt vectors, as well as their truncated versions. They also include the rational Witt vectors, the subring of the big Witt vectors of those power series with constant term one which are rational functions, which by a theorem of Almkvist [1, 2] is isomorphic to the cyclic K-theory ring. The following is the main result of §1.

Theorem A. – *Any PD-functor* $F \colon \mathrm{Ring} \to \mathrm{Ring}$ *extends canonically to an endofunctor on the category* $\mathrm{Ring}^{\mathrm{poly}}$ *of commutative rings and multiplicative polynomial laws. For any of the Witt vectors functors* W *listed above, this is the unique extension such that for any multiplicative polynomial law* $f \colon A \to B$ *the diagram*

$$\begin{tikzcd} W(A) \arrow[r, "W(f)"] \arrow[d, "w"'] & W(B) \arrow[d, "w"] \\ \prod A \arrow[r, "\prod f"] & \prod B \end{tikzcd}$$

commutes in $\mathrm{Ring}^{\mathrm{poly}}$, *where* w *is the ghost map of* W *and* $\prod f$ *is the product polynomial law.*

The theorem for a general PD-functor is proved in §1.3. We reduce the construction to torsion-free rings by means of a resolution argument. We then use that the n-homogeneous polynomial laws out of a torsion-free ring A are classified by the "*universal polynomial law*" $\gamma_A = (-)^{\otimes n} \colon A \to (A^{\otimes n})^{\Sigma_n}$, which we by definition send to the map

$$F(\gamma_A) \colon F(A) \xrightarrow{\gamma_{F(A)}} (F(A)^{\otimes n})^{\Sigma_n} \longrightarrow F(A^{\otimes n})^{\Sigma_n} \xleftarrow{\cong} F((A^{\otimes n})^{\Sigma_n}),$$

where the last map is an isomorphism by the axioms of a PD-functor. In fact we show that our extension of F to $\mathrm{Ring}^{\mathrm{poly}}$ is the unique one that sends γ_A to this map. When A has torsion, the universal polynomial law has value in the divided powers $\Gamma_n A$, which motivates the name PD-functor, where PD stands for "puissances divisées" (that is "divided powers"). In §1.4 we describe the ghost components of a polynomial law for the Witt vectors functors.

REMARK. – By a theorem of Almkvist [1, 2], our result shows that the cyclic K-group $\mathrm{K}_0^{\mathrm{cy}}(A)$, defined as K-group of the exact category of endomorphisms of finitely generated projective A-modules modulo the zero endomorphisms, is functorial in multiplicative polynomial laws. It is well known that $\mathrm{K}_0^{\mathrm{cy}}$ and K_0, as functors from additive categories, are functorial in polynomial *functors* (see [28] for a highly structured statement). It is however not clear how multiplicative polynomial laws of commutative rings relate to polynomial functors on the respective module categories. We also remark that K_0 is not a PD-functor (Example 1.15), and therefore that our theorem does not provide this extra functoriality for K_0.

In §2 we turn our attention to *n-polynomial maps*. These are the multiplicative maps $f: A \to B$ which satisfy the additive condition

$$(\mathrm{cr}_{n+1} f)(a_1, \dots, a_{n+1}) := \sum_{U \subset \{1,\dots,n+1\}} (-1)^{n+1-|U|} f(\sum_{l \in U} a_l) = 0.$$

As remarked above polynomial laws forget to polynomial maps, but this correspondence is neither surjective nor injective in general. However, it is bijective when the target ring is p-local and the degrees are at most $p-1$. By combining this observation with the theorem above we prove the following, in §2.2. For any integer or infinity $1 \leq m \leq \infty$, let $W_m(A; p)$ denote the ring of p-typical m-truncated Witt vectors.

COROLLARY B. – *The functor $W_m(-; p)$ extends to the partial category of multiplicative polynomial maps of degree at most $p-1$. That is, a multiplicative n-polynomial map $f: A \to B$ induces a multiplicative n-polynomial map*

$$W_m(f): W_m(A; p) \longrightarrow W_m(B; p)$$

for every $n < p$, with the property that if $f: A \to B$ and $g: B \to C$ are multiplicative and n and k-polynomial, respectively, and $nk < p$, then $W_m(g) \circ W_m(f) = W_m(g \circ f)$. This extension is unique with the property that the diagram

$$\begin{array}{ccc} W_m(A; p) & \xrightarrow{W_m(f)} & W_m(B; p) \\ \downarrow w & & \downarrow w \\ \prod_{j=0}^{m-1} A & \xrightarrow{\prod_{j=0}^{m-1} f} & \prod_{j=0}^{m-1} B \end{array}$$

commutes.

Much like the universal polynomials for the sum and multiplication of $W_m(A; p)$ it does not seem possible to give an explicit description of the Witt components of the map $W_m(f)$, but there is an inductive procedure for finding them. For odd p the first two components of $W_m(f)$ are

$$W_m(f)(a_0, a_1, \dots) = \big(f(a_0), \sum_{i=1}^{p-1} (-1)^i (\binom{p}{i}/p) f(a_0^p + i a_1), \dots\big)$$

(see Example 2.9). When f is a ring homomorphism, one can verify using standard binomial identities that the second component is equal to $f(a_1)$, so that this construction indeed extends the usual functoriality of $W_m(-; p)$ in ring homomorphisms. In §2.2 we also discuss