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LEVEL OF DISTRIBUTION OF UNBALANCED CONVOLUTIONS

BY ÉTIENNE FOUVRY AND MAKSYM RADZIWIŁŁ

ABSTRACT. – We show that if an essentially arbitrary sequence supported on an interval containing x integers, is convolved with a tiny Siegel-Walfisz-type sequence supported on an interval containing $\exp((\log x)^{\varepsilon})$ integers then the resulting multiplicative convolution has (in a weak sense) level of distribution $x^{1/2+1/66-\varepsilon}$ as x goes to infinity. This dispersion estimate has a number of consequences for: the distribution of the kth divisor function to moduli $x^{1/2+1/66-\varepsilon}$ for any integer $k \geq 1$, the distribution of products of exactly two primes in arithmetic progressions to large moduli, the distribution of sieve weights of level $x^{1/2+1/66-\varepsilon}$ to moduli as large as $x^{1-\varepsilon}$ and for the average distribution of sums of two squares for almost all moduli q of size $x^{1-\varepsilon}$. Our result improves and is inspired by earlier work of Green (and subsequent work of Granville-Shao) which is concerned with the distribution of 1-bounded multiplicative functions in arithmetic progressions to large moduli. As in these previous works the main technical ingredients are the recent estimates of Bettin-Chandee for trilinear forms in Kloosterman fractions and the estimates of Duke-Friedlander-Iwaniec for bilinear forms in Kloosterman fractions.

RÉSUMÉ. – Nous prouvons que, si une suite essentiellement arbitraire, de support inclus dans un intervalle contenant x entiers, est convolée à une suite courte de type Siegel-Walfisz, de support inclus dans un intervalle contenant $\exp((\log x)^{\varepsilon})$ entiers, le résultat obtenu par cette convolution multiplicative, est une suite dont le niveau de répartition est, au sens faible, en $x^{1/2+1/66-\varepsilon}$, pour x tendant vers l'infini. Cette estimation de variance a plusieurs conséquences : l'équirépartition, pour k quelconque, de la fonction nombre de diviseurs d'ordre k jusqu'aux modules en $x^{1/2+1/66-\varepsilon}$, l'équiréparition des produits de deux nombres premiers dans les progressions arithmétiques de grands modules, l'équirépartition des poids du crible de niveau en $x^{1/2+1/66-\varepsilon}$ dans des modules allant jusqu'à $x^{1-\varepsilon}$, enfin, la répartition en moyenne des entiers sommes de deux carrés, dans des progressions arithmétiques de modules allant jusqu'à $x^{1-\varepsilon}$. Les améliorations qu'apporte notre travail, s'inspirent d'un article de Green (et d'un résultat ultérieur de Granville-Shao) qui traitent de la répartition des fonctions multiplicatives, bornées par 1, dans les progressions arithmétiques de grand module. Comme dans les travaux antérieurs le principal ingrédient technique est la récente majoration de Bettin–Chandee des formes trilinéaires en fractions de Kloosterman, dans le prolongement de la majoration de Duke–Friedlander–Iwaniec des formes bilinéaires de ces mêmes fractions.

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1. Introduction

A major open problem in prime number theory is to show the existence of some $\delta > 0$ such that for any integer $a \neq 0$ and for any $A > 0$ we have

$$\sum_{\substack{q \leq x^{1/2+\delta} \\ (q,a)=1}} \Big| \sum_{\substack{p \leq x \\ p \equiv a \pmod q}} 1 - \frac{1}{\varphi(q)} \sum_{\substack{p \leq x \\ (p,q)=1}} 1 \Big| \ll_{a,A} x(\log x)^{-A} \tag{1}$$

uniformly for $x \geq 2$. We would then say that *primes have a level of distribution* $x^{1/2+\delta}$ *in a weak sense*, and call $\frac{1}{2}+\delta$ an *exponent of distribution of the primes in a weak sense*. If we could establish a similar statement but with the maximum over $(a,q)=1$ inside the sum over q we would then drop the *weak* adjective (see [12] for a precise definition). For brevity we will not further distinguish between the two terms since we will be only concerned with the former "weak sense".

By Chebyschev's inequality (1) implies that for "almost all" (i.e., all with the exception of a density zero subset) moduli $q \leq x^{1/2+\delta}$ the primes are well-distributed in arithmetic progressions $n \equiv a \pmod q$.

The inequality (1) follows for $\delta < 0$ from the Bombieri-Vinogradov theorem, and for $\delta = 0$ and $A < 2$ from work of Bombieri-Friedlander-Iwaniec [4]. Zhang [19] established (1) for some $\delta > 0$ with q restricted to x^{ε}-smooth moduli (see also [5]). The problem of establishing (1) for some $\delta > 0$ is challenging since it lies beyond the capability of the Generalized Riemann Hypothesis.

Underpinning any current approach to (1) are *dispersion estimates* originally invented by Linnik. Roughly a dispersion estimate asserts that for $M, N \geq 1$ and two arbitrary sequences $\boldsymbol{\alpha} = (\alpha_m)_{M<m\leq 2M}$ and $\boldsymbol{\beta} = (\beta_n)_{N<n\leq 2N}$ of complex numbers satisfying some minor technical conditions, we have for any $a \neq 0$ fixed,

$$\sum_{\substack{Q \leq q \leq 2Q \\ (q,a)=1}} \Big| \sum_{\substack{x<mn\leq 2x \\ mn \equiv a \pmod q}} \alpha_m \beta_n - \frac{1}{\varphi(q)} \sum_{\substack{x<mn\leq 2x \\ (mn,q)=1}} \alpha_m \beta_n \Big| \ll_{a,A} x(\log x)^{-A} \tag{2}$$

for $x \asymp MN$, uniformly in $Q \leq x^{1/2+\delta}$ for some $\delta > 0$. As usual the case of $\delta < 0$ falls within the scope of techniques related to the Bombieri-Vinogradov theorem, and is well-understood (see [3, Theorem 0] or [14, Theorem 9.16]).

A necessary assumption in a dispersion estimate is that at least one of the sequences is well-distributed in arithmetic progressions having small moduli. This is referred to as a *Siegel-Walfisz condition*.

DEFINITION 1. – *We say that a sequence $\boldsymbol{\beta} = (\beta_n)$ of complex numbers satisfies a Siegel-Walfisz condition (alternatively we also say that $\boldsymbol{\beta}$ is Siegel-Walfisz), if there exists an integer $k > 0$ such that for any fixed $A > 0$, uniformly in $x \geq 2$, $q > |a| \geq 1, r \geq 1$ and $(a,q)=1$, we have,*

$$\sum_{\substack{x<n\leq 2x \\ n \equiv a \pmod q \\ (n,r)=1}} \beta_n = \frac{1}{\varphi(q)} \sum_{\substack{x<n\leq 2x \\ (n,qr)=1}} \beta_n + O_A(\tau_k(r) \cdot x(\log x)^{-A}),$$

where $\tau_k(n)$ is the k-th divisor function $\tau_k(n) := \sum_{n_1 \ldots n_k = n} 1$. Throughout the paper the implicit constant in $O(\cdot)$ is allowed to depend on k.

For $\delta > 0$ there are few results that address (2) in wide generality. As we already mentioned at least one of the sequence $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ needs to be Siegel-Walfisz. In all the cases that are known (i.e., [3, Theorem 3], [8, Théorème 1] and [9, Corollaire 1]) the Siegel-Walfisz sequence needs to be supported on an interval of length at least $x^{\varepsilon} \cdot (Q/\sqrt{x} + 1)^2$ (and no longer than say $x^{1/6-\varepsilon}$ or $x^{1/12-\varepsilon}$). In particular the length of this interval is at least a power of x as soon as Q increases beyond $\sqrt{x}$ by a small power of x.

Our first result is a new dispersion estimate that roughly shows that (2) can be obtained with $Q = x^{1/2+1/66-\varepsilon}$ even if the Siegel-Walfisz sequence $\boldsymbol{\beta}$ is supported on a tiny interval of length $\exp((\log x)^{\varepsilon})$ for any sufficiently small $\varepsilon > 0$. We find this rather striking, since this means that a tiny smoothing of an otherwise arbitrary sequence supported on x integers allows one to suddenly reach a level of distribution $x^{1/2+1/66-\varepsilon}$. We call such a convolution of two sequences of drastically different sizes an *unbalanced convolution*.

COROLLARY 1.1. – *Let $k > 0$ and $\varepsilon > 0$ be given. Let $\boldsymbol{\alpha} = (\alpha_m)_{M<m\leq 2M}$ and $\boldsymbol{\beta} = (\beta_n)_{N<n\leq 2N}$ be two sequences of complex numbers such that $|\alpha_m| \leq \tau_k(m)$ and $|\beta_n| \leq \tau_k(n)$ for all $m, n \geq 1$. Suppose that $\boldsymbol{\beta}$ is Siegel-Walfisz. Then for every $A > 0$, uniformly in $M, N \geq 2$ with $MN/2 \leq x \leq 4MN$ we have,*

$$\sum_{\substack{Q\leq q\leq 2Q \\ (q,a)=1}} \Bigg| \sum_{\substack{x<mn\leq 2x \\ mn\equiv a \pmod q}} \alpha_m \beta_n - \frac{1}{\varphi(q)} \sum_{\substack{x<mn\leq 2x \\ (mn,q)=1}} \alpha_m \beta_n \Bigg| \ll_A x(\log x)^{-A} \tag{3}$$

provided that any of the following three conditions holds

(i) $\exp((\log x)^{\varepsilon}) \leq N \leq Q^{-11/12} \cdot x^{17/36-\varepsilon}$ *and* $1 \leq |a| \leq x/12$.

(ii) $\exp((\log x)^{\varepsilon}) \leq N \leq x^{7/90-\varepsilon}$, $Q \leq x^{53/105-\varepsilon}$ *and* $1 \leq |a| \leq x/12$.

(iii) $\exp((\log x)^{\varepsilon}) \leq N \leq x^{101/630-\varepsilon}$, $Q \leq x^{53/105-\varepsilon}$ *and* $1 \leq |a| \leq (x/4)^{\varepsilon/1000}$.

Note that the left-hand side of (3) is identically zero if x falls outside of the interval $[MN/2, 4MN]$. Here i) gives the strongest estimate in the Q-aspect for very small N, allowing for Q to go up to $x^{1/2+1/66-3\varepsilon}$ provided that $N \leq x^{\varepsilon}$, whereas (ii) and (iii) give stronger uniformity in the N-aspect at the price of a slightly weaker level of distribution.

A numerically stronger, but conditional, version of Corollary 1.1 appears in Fouvry's thesis [7]. Fouvry's result depends on the assumption of the still unproven Hooley's $R^{\star}$-conjecture on cancelations in short incomplete Kloosterman sums. To obtain the unconditional Corollary 1.1 we appeal instead to results of Duke-Friedlander-Iwaniec [6] and Bettin-Chandee [1]. These results can be used as unconditional substitutes for Hooley's $R^{\star}$-conjecture "on average". A similar observation is implicit in the recent work of Green [16] which is the second starting point for our work.

Our dispersion estimate has a number of interesting corollaries, many of them relying on the observation that most integers n can be factored as $n = pm$ with p a small prime in the range $[\exp((\log x)^{\varepsilon}), x^{\varepsilon}]$. The first corollary concerns the distribution of the kth divisor function in arithmetic progressions to large moduli.