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THE DEGREE OF IRRATIONALITY OF MOST ABELIAN SURFACES IS 4

BY Olivier MARTIN

Abstract. – The degree of irrationality of a smooth projective variety X is the minimal degree of a dominant rational map $X \dashrightarrow \mathbb{P}^{\dim X}$. We show that if an abelian surface A over $\mathbb{C}$ is such that the image of the intersection pairing $\mathrm{Sym}^2 NS(A) \to \mathbb{Z}$ does not contain 12, then it has degree of irrationality 4. In particular, a very general $(1,d)$-polarized abelian surface has degree of irrationality 4 provided that $d \nmid 6$. This answers two questions of Yoshihara by providing the first examples of abelian surfaces with degree of irrationality greater than 3 and showing that the degree of irrationality is not isogeny-invariant for abelian surfaces.

Résumé. – Le degré d'irrationnalité d'une variété projective lisse X est le degré minimal d'une application rationnelle dominante $X \dashrightarrow \mathbb{P}^{\dim X}$. Nous montrons que si une surface abélienne A sur $\mathbb{C}$ est telle que l'image de l'accouplement d'intersection $\mathrm{Sym}^2 NS(A) \to \mathbb{Z}$ ne contient pas le nombre 12, alors son degré d'irrationnalité est 4. En particulier, le degré d'irrationnalité d'une surface abélienne très générale munie d'une polarization de type $(1,d)$ est 4 si $d \nmid 6$. Ce résultat nous permet de répondre à deux questions de Yoshihara en fournissant les premiers exemples de surfaces abéliennes dont le degré d'irrationnalité est supérieur à 3 et en montrant que le degré d'irrationnalité n'est pas invariant sous isogénie pour les surfaces abéliennes.

1. Introduction

The *degree of irrationality* $\mathrm{irr}(X)$ of a smooth projective variety X is the minimal degree of a dominant rational map $X \dashrightarrow \mathbb{P}^{\dim X}$. It is a birational invariant that measures how far X is from being rational. While we have a relatively good understanding of the degree of irrationality of very general hypersurfaces of large degree (see [3]), comparatively little is known for other classes of complex projective varietes. Since [3] exploits positivity properties of K_X to provide lower bounds on measures of irrationality for X, it is natural to consider

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K-trivial varieties as a case of particular interest. In this direction, [9],[2],[12],[7], and [6] use rational equivalence of zero-cycles on abelian varieties to obtain lower bounds on measures of irrationality for very general abelian varieties. These articles all use some form of induction on the dimension of the abelian variety, with abelian surfaces as base case. Consequently, they are powerless to provide bounds on measures of irrationality for abelian surfaces.

The degree of irrationality of any abelian surface is at least 3 by a result of Alzati and Pirola [1]. Tokunaga and Yoshihara also proved that this bound is sharp in [11]. They show that if an abelian surface contains a smooth curve of genus 3, then it admits a degree 3 dominant rational map to $\mathbb{P}^2$. In particular, a very general $(1,2)$-polarized abelian surface has degree of irrationality 3. For the same reason, the degree of irrationality of the product of two isogenous non-CM elliptic curves E and E' is 3 if the minimal degree of an isogeny between E and E' is not $1, 3, 5, 9, 11, 15, 21, 29, 35, 39, 51, 65, 95, 105, 165, 231$, or one other odd number if the generalized Riemann hypothesis is false [10]. Tokunaga and Yoshihara also provide an example of an elliptic curve E with complex multiplication such that $E \times E$ contains a smooth genus 3 curve, as well as an example of a Jacobian of a genus 2 curve which has degree of irrationality 3.

These results prompted Yoshihara to ask the following questions in Problem 10 of [13]:

1. Is there an abelian surface with degree of irrationality greater than 3?
2. Do isogenous abelian surfaces have the same degree of irrationality?

The most recent progress on the study of the degree of irrationality of very general abelian surfaces is due to Chen who showed in [4] that it is at most 4, independently of the polarization type. Chen's result was improved on in [5], where it is shown that this bound applies to every abelian surface. The main result of this article is:

THEOREM 1.1. – *If A is an abelian surface such that the image of the intersection pairing $\mathrm{Sym}^2 NS(A) \to \mathbb{Z}$ does not contain $12\mathbb{Z}$, then the degree of irrationality of A is 4.*

COROLLARY 1.2. – *A very general $(1,d)$-polarized abelian surface has degree of irrationality 4 if $d \nmid 6$.*

Proof. – A very general abelian variety A has Picard number 1 so that $NS(A)$ is generated by the polarizing class. The image of $\mathrm{Sym}^2 NS(A) \to \mathbb{Z}$ is thus $2d\mathbb{Z}$, where d is the degree of the polarization. □

This theorem answers the first question of Yoshihara affirmatively and the second question negatively. Indeed, a very general $(1,2)$-polarized abelian surface has degree of irrationality 3 and is isogenous to a $(1,d)$-polarized abelian surface of Picard number 1 for some $d \nmid 6$. It follows that the degree of irrationality is not an isogeny invariant for abelian surfaces.

REMARK 1.3. – These results also hold if the degree of irrationality is replaced with the minimal degree of a dominant rational map to a surface S with $CH_0(S) \cong \mathbb{Z}$. Indeed, we only use that $CH_0(\mathbb{P}^2) \cong \mathbb{Z}$ and that

$$[\Delta_{\mathbb{P}^2}] \in [\mathrm{pt} \times \mathbb{P}^2] + [\mathbb{P}^2 \times \mathrm{pt}] + \mathrm{Sym}^2 NS(\mathbb{P}^2) \subset H^4(\mathbb{P}^2 \times \mathbb{P}^2, \mathbb{Z}),$$

which is also valid for surfaces with a trivial Chow group of zero-cycles.

These results motivate the following questions.

QUESTIONS 1.4. – 1. *What is the degree of irrationality of very general* $(1,1)$*,* $(1,3)$*, and* $(1,6)$*-polarized abelian surfaces?*

2. *If E and E' are non-isogenous elliptic curves, what is the degree of irrationality of $E \times E'$?*

The second question was suggested by Yoshihara as a possible approach to finding abelian surfaces with degree of irrationality 4. It is generally believed that the answer is 4. Unfortunately our approach fails to answer this question.

Theorem 1.1 is obtained by a simple cohomological computation. Although Mumford's theorem for rational equivalence of zero-cycles on surfaces with $p_g \neq 0$ makes an appearance, our methods are completely different from those of [9],[2],[12], [7], and [6]. In the second and final section we offer a proof of Theorem 1.1.

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2. Proofs

Proof of Theorem 1.1. – By the Alzati-Pirola bound from [1] and the results of [4] and [5], it suffices to show that if there is a dominant rational map of degree 3

$$\varphi : A \dashrightarrow \mathbb{P}^2,$$

then 12 lies in the image of $\mathrm{Sym}^2 NS(A) \to \mathbb{Z}$. Such a rational map gives rise to another rational map $F_\varphi : \mathbb{P}^2 \dashrightarrow \mathrm{Sym}^3(A)$, which associates to a generic point of $\mathbb{P}^2$ the fiber of φ over it. We will denote by Z' its image. Note that Z' is a *constant cycle subvariety*: it parametrizes rationally equivalent effective zero-cycles of degree 3 on A. We can thus assume that $Z' \subset \mathrm{Sym}^{3,0}(A)$, where $\mathrm{Sym}^{3,0}(A)$ is the fiber of the summation map $\mathrm{Sym}^3(A) \to A$ over $0_A \in A$. Let

$$q : A^{3,0} := \ker(A^3 \to A) \to \mathrm{Sym}^{3,0}(A)$$

be the quotient map and $Z := q^{-1}(Z')$.

We fix the isomorphism $\iota : A^2 \to A^{3,0}$ given by $\iota(a,b) = (a, b, -a-b)$ and we abuse notation to write Z for $\iota^{-1}(Z)$. Letting $U \subset A$ be an open on which the rational map φ restricts to an étale morphism, we have

$$Z = \overline{\{(x,y) \in U^2 : x \neq y, \varphi(x) = \varphi(y)\}}.$$

Another way to obtain this cycle is by resolving the indeterminacies of φ to get a morphism $\widetilde{\varphi} : \widetilde{A} \to \mathbb{P}^2$ and a birational morphism $\pi : \widetilde{A} \to A$, such that $\varphi|_U = \widetilde{\varphi} \circ (\pi|_U)^{-1}$.