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ASYMPTOTIC COUNTING OF MINIMAL SURFACES AND OF SURFACE GROUPS IN HYPERBOLIC 3-MANIFOLDS
[according to Calegari, Marques and Neves]

by **François Labourie**

To Srishti Dhar Chatterji,
who attended Séminaire Bourbaki for most of half a century

INTRODUCTION

The study of the geodesic flow in closed negatively curved manifolds is a beautiful mix of topology, Riemannian geometry, geometric group theory and ergodic theory. We know in this situation that closed geodesics are in one-to-one correspondence with conjugacy classes of elements of the fundamental group, or equivalently, with the set of homotopy classes of maps of circles in the manifold. Even though closed geodesics are infinite in number, we have a good grasp — thanks to the notion of *topological entropy* — of how the number of these geodesics grows with respect to the length. We also have a computation of this topological entropy in hyperbolic spaces by Bowen (1972) and Margulis (1969) and rigidity results for this entropy by Besson, Courtois, and Gallot (1995) and Hamenstädt (1990).

While the statements of this first series of results seem to deal only with closed geodesics, the foliation of the unit tangent bundle by orbits of the geodesic flow plays a fundamental role. The study of invariant measures by the geodesic flow is a crucial tool, and the equidistribution of closed geodesics by Bowen (1972) and Margulis (1969) for hyperbolic manifolds a central result. We refer to section 1 for more precise definitions, results and references.

For many reasons — as we discuss in section 2 — closed totally geodesic submanifolds of dimension at least 2 are quite rare. However, in constant curvature, the foliation of the Grassmannian of k-planes coming from totally geodesic planes is a natural generalization of the geodesic flow and several crucial results of Ratner (1991a,b) and Shah (1991) as well as McMullen, Mohammadi, and Oh (2017) describe closed invariant sets and invariant measures. This foliation stops to make sense in variable curvature, at least far away from the constant curvature situation, although for metrics close to hyperbolic ones, a result by Gromov (1991a) — see also Lowe, 2020

— shows that the foliation of the Grassmann bundle persists when one replaces totally geodesic submanifolds by minimal ones.

If we move in the topological direction, going from circles to surfaces, Kahn–Marković Surface Subgroup Theorem (Kahn and Marković, 2012b) provides the existence of many surface subgroups in the fundamental group of a hyperbolizable 3-manifold M. A subsequent result of Kahn and Marković (2012a) gives an asymptotic of the number of these surface groups with respect to the genus — see Theorem 3.8.

However this asymptotic counting does not involve the underlying Riemannian geometry as opposed to the topological entropy that we discussed in the first paragraph. The next step is to use fundamental results of Schoen and Yau (1979) and Sacks and Uhlenbeck (1982), which tells us that every such surface group can be realized by a minimal surface — although non necessarily uniquely.

In Calegari, Marques, and Neves (2020), the authors propose a novel idea: count asymptotically with respect to the area these minimal surfaces, but when the boundary at infinity of those minimal surfaces becomes more and more circular, or more precisely are K-quasicircles, with K approaching 1. The precise definition of this counting requires the description of quasi-Fuchsian groups and their boundary at infinity, done in section 3, and their main result (Theorem 6.1) is presented in section 6. These results define an entropy-like constant $E(M, h)$ for minimal surfaces in a Riemannian manifold (M, h) of curvature less than -1. The main result of Calegari, Marques, and Neves (2020) is to compute it for hyperbolic manifolds, gives bounds in the general case and most notably proves a rigidity result: $E(M, h) = 2$ if and only if h is hyperbolic. Altogether, these results mirror those for closed geodesics.

When one moves to studying solution of elliptic partial differential equations, for instance minimal surfaces or pseudo-holomophic curves, the situation is different from the chaotic behavior of the geodesic flow. While there is a huge literature about moduli spaces of solutions when one imposes constraints such as homology classes, we do not have that many results describing a moduli space of all solutions: possibly immersed with dense images, in other words to continue the process for minimal surfaces described in the introduction of Gromov (1991a) for geodesics: *if one wishes to understand closed geodesics not as individuals but as members of a community one has to look at all (not only closed) geodesics in X which form an* 1*-dimensional foliation of the projectivized tangent bundle.*

The presentation of these notes shifts around the ideas used in Calegari, Marques, and Neves (2020) and follows more directly the philosophy introduced in Gromov (1991a). We focus on the construction of such a moduli space — that we call the *phase space of stable minimal surfaces* — and its topological properties — see section 5.1 and Theorem 5.2. These properties are a rephrasing of Theorem 4.18 about quasi-isometric properties of stable minimal surfaces, relying on results of Seppi (2016) and a "Morse type Lemma" argument by Calegari, Marques, and Neves (2020, Theorem 3.1). This space is the analogue, in our situation, of the geodesic flow and the $\mathbb{R}$-action is replaced by an $\mathsf{SL}_2(\mathbb{R})$-action.

Then we move to studying $\mathrm{SL}_2(\mathbb{R})$-invariant measures on this phase space and show they are related to what we call *laminar currents* which are the analogues in our situation of geodesic currents — see Bonahon (1997). The main result is now an equidistribution result in this situation: Theorem 1.3. This theorem follows from the techniques of the proof of Surface Subgroup Theorem using the presentation given in Kahn, Labourie, and Mozes (2018).

This Equidistribution Theorem and the construction of the phase space allows, by comparing the countings with respect to the area and to the genus — as in Kahn and Marković, 2012a — to proceed quickly to the proof of the results of Calegari, Marques, and Neves (2020) when, for the rigidity result, we assume that h is close enough to a hyperbolic metric.

The whole article of Calegari, Marques, and Neves (2020) mixes beautiful ideas from many subjects, adding to the mix of topology, Riemannian geometry, geometric group theory and ergodic theory used in the study of the geodesic flow, a pinch of geometric analysis. The approach given in these notes is not just to present the proof but also to take the opportunity to tour some of the fundamental results in these various mathematics. [1] We take some leisurely approach and explain some of the main results and take the time to give a few simple proofs and elementary discussions: the clever proof of Thurston showing that there are only finitely many surface groups of a given genus in the fundamental group of a hyperbolic manifold, the discussion of stable minimal surfaces, the geometric analysis trick that derives from a rigidity result (here the characterization of the plane as the unique stable minimal surface in $\mathbb{R}^3$) some compactness results (Proposition 4.9).

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1. COUNTING GEODESICS AND EQUIDISTRIBUTION

When (M, h) is a negatively curved manifold, there is a one-to-one correspondence between conjugacy classes of elements of $\pi_1(M)$ and closed geodesics. Even though there are infinitely many closed geodesics, we can count them "asymptotically". Equivalently, this will give an asymptotic count of the conjugacy classes of elements of $\pi_1(M)$, or to start a point of view that we shall pursue later, the set of free homotopy classes of maps of S^1 in M.

We review here some important results that will be useful in our discussion and serve as a motivation.

1. The introduction of Calegari, Marques, and Neves (2020) also addresses minimal hypersurfaces in higher dimension that we do not discuss here

1.1. Entropy and asymptotic counting of geodesics

Let (M, h) be a closed manifold of negative curvature. Fixing a positive constant T, there are only finitely many closed geodesics of length less than T. Let us define

$$\Gamma_h(T) := \{\text{geodesic } \gamma \ ; \ \mathrm{length}(\gamma) \leqslant T\}.$$

The following limit, when it is defined, is called the *topological entropy of M*:

$$\mathrm{h}_{\mathrm{top}}(M, h) := \lim_{t\to\infty} \frac{1}{T} \log\left(\sharp\Gamma_h(T)\right).$$

We will see it is always defined in negative curvature. It measures the exponential growth of the number of geodesics with respect to the length.

The topological entropy is related to the *volume entropy of M* defined by

$$\mathrm{h}_{\mathrm{vol}}(M, h) = \liminf_{t\to\infty} \frac{1}{R} \log\bigl(\mathrm{Vol}(B(x, R))\bigr),$$

where $B(x, R)$ is the ball of radius R in the universal cover $\widetilde{M}$ of M, x any point in $\widetilde{M}$. The volume entropy does not depend on the choice of the point x and we have

Theorem 1.1. — *Let (M, h) be a closed negatively curved manifold.*

1) *The topological* $\mathrm{h}_{\mathrm{top}}(M, h)$ *is well-defined. When h_0 is hyperbolic,* [2]

$$\mathrm{h}_{\mathrm{top}}(M, h_0) = \dim(M) - 1.$$

2) *We have*

$$\mathrm{h}_{\mathrm{top}}(M, h) = \mathrm{h}_{\mathrm{vol}}(M, h) = \lim_{R\to\infty} \frac{\log\left(\sharp\{\gamma \in \pi_1(M) \ ; \ d_M(\gamma \cdot x, x) \leqslant R\}\right)}{R}.$$

The first item is a celebrated result by Bowen (1972) and Margulis (1969). The second item is due to Manning (1979).

1.1.1. *Rigidity of the entropy.* — We have several rigidity theorems for the entropy. First, if a metric on closed manifold has curvature less than -1, then

$$\mathrm{h}_{\mathrm{vol}}(M, h) \geqslant \dim(M) - 1,$$

with equality if and only if h is hyperbolic. For deeper results in the presence of upper bounds on the curvature, see Pansu (1989) and Hamenstädt (1990). As a special case of Besson, Courtois, and Gallot (1995), we have, when we drop the condition on the curvature

Theorem 1.2. — *Let (M, h_0) be a hyperbolic manifold of dimension m and h another metric on M, then*

$$\mathrm{h}_{\mathrm{vol}}(M, h)^m \cdot \mathrm{Vol}(M, h) \geqslant \mathrm{h}_{\mathrm{vol}}(M, h_0)^m \cdot \mathrm{Vol}(M, h_0).$$

The equality implies that h has constant curvature.

In this exposé, we will only use the case of $m = 2$, which is due to Katok (1982).

2. That is when the curvature is constant and equal to -1

1.2. Equidistribution

This asymptotic counting has a counterpart called *equidistribution*. Let us first recall that geodesics are solutions of some second order differential equation, and we may as well consider non closed geodesics in the Riemannian manifold M. Let us consider the *phase space* $\mathcal{G}$ of this equation as the space of maps γ from $\mathbb{R}$ to M, where γ is an arc length parametrized solution of the equation. The precomposition by translation gives a right action by $\mathbb{R}$, and thus $\mathcal{G}$ is partitioned into *leaves* which are orbits of the right action of $\mathbb{R}$. The space $\mathcal{G}$ canonically identifies with the unit tangent bundle $\mathsf{U}M$ by the map $\gamma \mapsto (\gamma(0), \dot{\gamma}(0))$, and the above $\mathbb{R}$-action corresponds to the action of the *geodesic flow*.

We may thus associate to each closed orbit γ of length ℓ a unique probability measure δ_γ on $\mathcal{G} = \mathsf{U}M$ supported on γ, $\mathbb{R}$-invariant and so that for any function on $\mathsf{U}M$

$$\int_{\mathsf{U}M} f \,\mathrm{d}\delta_\gamma := \frac{1}{\ell}\int_0^\ell f\big(\gamma(s)\big)\,\mathrm{d}s.$$

When M is hyperbolic, another natural and $\mathbb{R}$-invariant probability measure comes from the left invariant μ_{Leb} measure (under the group of isometries) in the universal cover.

The next result is intimately related to Theorem 1.1 and also due to Bowen (1972) and Margulis (1969).

Theorem 1.3. — *Assume (M, h_0) is hyperbolic, then*

$$\lim_{T\to\infty} \frac{1}{\sharp\Gamma_{h_0}(T)} \sum_{\gamma\in\Gamma_{h_0}(T)} \delta_\gamma = \mu_{Leb}.$$

2. TOTALLY GEODESIC SUBMANIFOLDS OF HIGHER DIMENSION

As a first attempt of generalization, it is quite tempting to understand what happens to *totally geodesic* submanifolds of higher dimension, where by totally geodesic we mean complete and such that any geodesic in the submanifold is a geodesic for the ambient manifold.

2.1. Closed totally geodesic submanifolds are rare

One easily constructs by arithmetic means hyperbolic manifolds with infinitely many closed totally geodesic submanifolds, however this situation is exceptional and we have, as a special case of a beautiful recent theorem by Margulis and Mohammadi (2019) — generalized in Bader, Fisher, Miller, and Stover (2021):

Theorem 2.1. — *If a closed hyperbolic 3-manifold M contains infinitely many closed totally geodesic subspaces of dimension at least 2, then M is arithmetic.*