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ABOUT PLANE PERIODIC WAVES OF THE NONLINEAR SCHRÖDINGER EQUATIONS

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ABSTRACT. — The present contribution contains a quite extensive theory for the stability analysis of plane periodic waves of general Schrödinger equations. On the one hand, we put the one-dimensional theory, or in other words the stability theory for longitudinal perturbations, on par with the one available for systems of Korteweg type, including results on coperiodic spectral instability, nonlinear coperiodic orbital stability, sideband spectral instability and linearized large-time dynamics in relation with modulation theory, and resolutions of all the involved assumptions in both the small-amplitude and large-period regimes. On the other hand, we provide extensions of the spectral part of the latter to the multidimensional context. Notably, we provide suitable multidimensional modulation formal asymptotics, validate those at the spectral level, and use them to prove that waves are always spectrally unstable in both the small-amplitude and the large-period regimes.

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RÉSUMÉ (*À propos des ondes planes périodiques des équations de Schrödinger non linéaires*). — Le travail présenté ici comprend une théorie relativement complète permettant l'analyse de la stabilité des ondes planes périodiques des équations de Schrödinger générales. D'une part, nous mettons la théorie unidimensionnelle, ou autrement dit la théorie de stabilité sous perturbations longitudinales, au niveau de celle disponible pour les systèmes de type Korteweg, en y incluant des résultats sur l'instabilité spectrale co-périodique, la stabilité orbitale non linéaire co-périodique, l'instabilité spectrale latérale et la dynamique linéarisée en temps long et ses relations avec la théorie de la modulation, et en résolvant toutes les hypothèses associées dans les régimes de petite amplitude et de grande période. D'autre part, nous étendons la partie spectrale de cette analyse au contexte multidimensionnel. En particulier, nous développons une asymptotique formelle de modulation multidimensionnelle, validons celle-ci au niveau spectral et l'utilisons pour démontrer que les ondes sont toujours spectralement instables à la fois dans les régimes de petite amplitude et de grande période.

1. Introduction

We consider Schrödinger equations in the form

$$\mathrm{i}\,\partial_t f = -\operatorname{div}_{\mathbf{x}}\left(\kappa(|f|^2)\,\nabla_{\mathbf{x}} f\right) + \kappa'(|f|^2)\,\|\nabla_{\mathbf{x}} f\|^2\, f + 2\,W'(|f|^2)\, f \tag{1}$$

(or some anisotropic generalizations) with W a smooth real-valued function and κ a smooth positive-valued function, bounded away from zero, where the unknown f is complex valued, $f(t,\mathbf{x}) \in \mathbf{C}$, and $(t,\mathbf{x}) \in \mathbf{R}\times\mathbf{R}^d$. Note that the sign assumption on κ may be replaced with the assumption that κ is real valued and far from zero since one may change the sign of κ by replacing (f,κ,W) with $(\overline{f},-\kappa,-W)$.

Since the nonlinearity is not holomorphic in f, it is convenient to adopt a real point of view and introduce real and imaginary parts $f = a + \mathrm{i}\, b$, $\mathbf{U} = \begin{pmatrix} a \\ b \end{pmatrix}$. Multiplication by $-\mathrm{i}$ is, thus, encoded in

$$\mathbf{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{2}$$

and Equation (1) takes the form

$$\partial_t \mathbf{U} = \mathbf{J}\left(-\operatorname{div}_{\mathbf{x}}\left(\kappa(\|\mathbf{U}\|^2)\,\nabla_{\mathbf{x}}\mathbf{U}\right) + \kappa'(\|\mathbf{U}\|^2)\,\|\nabla_{\mathbf{x}}\mathbf{U}\|^2\,\mathbf{U} + 2\,W'(\|\mathbf{U}\|^2)\,\mathbf{U}\right). \tag{3}$$

The problem has a Hamiltonian structure

$$\partial_t \mathbf{U} = \mathbf{J}\,\delta\mathscr{H}_0[\mathbf{U}] \qquad \text{with} \qquad \mathscr{H}_0\,[\mathbf{U}] = \tfrac{1}{2}\kappa(\|\mathbf{U}\|^2)\|\nabla_{\mathbf{x}}\mathbf{U}\|^2\,, + W(\|\mathbf{U}\|^2),$$

with δ denoting the variational gradient[1]. Indeed, our interest in (1) originates in the fact that we regard the class of equations (1) as the most natural class of

1. See the notational section at the end of the present Introduction for a definition.

isotropic quasi/linear dispersive Hamiltonian equations, including most classical semilinear Schrödinger equations. See [54] for a comprehensive introduction to the latter. In Appendix C, we also show how to treat some anisotropic versions of the equations.

Note that in the above form, invariances are embedded with respect to rotations, time translations and space translations; if f is a solution, so is $\widetilde{f}$ when

$$\begin{aligned}
\widetilde{f}(t,\mathbf{x}) &= \mathrm{e}^{-\,\mathrm{i}\,\phi_0}\, f(t,\mathbf{x})\,, & \phi_0 &\in \mathbf{R}\,, & &\text{rotational invariance}\,,\\
\widetilde{f}(t,\mathbf{x}) &= f(t-t_0,\mathbf{x})\,, & t_0 &\in \mathbf{R}\,, & &\text{time translation invariance}\,,\\
\widetilde{f}(t,\mathbf{x}) &= f(t,\mathbf{x}-\mathbf{x}_0)\,, & \mathbf{x}_0 &\in \mathbf{R}^d\,, & &\text{space translation invariance}\,.
\end{aligned}$$

Actually, rotations and time and space translations leave the Hamiltonian $\mathscr{H}_0$ essentially unchanged, in a sense made explicit in Appendix A. Thus, through a suitable version of Noether's principle, they are associated with conservation laws, respectively on mass $\mathscr{M}[\mathbf{U}] = \frac{1}{2}\|\mathbf{U}\|^2$, Hamiltonian $\mathscr{H}_0[\mathbf{U}]$ and momentum $\mathbb{Q}[\mathbf{U}] = (\mathbb{Q}_j[\mathbf{U}])_j$, with $\mathbb{Q}_j[\mathbf{U}] = \frac{1}{2}\mathbf{J}\mathbf{U}\cdot\partial_j\mathbf{U}$, $j = 1,\dots,d$. Namely, invariance by rotation implies that any solution $\mathbf{U}$ to (3) satisfies the mass conservation law

$$\partial_t\mathscr{M}(\mathbf{U}) = \sum_j \partial_j\left(\mathbf{J}\delta\mathscr{M}[\mathbf{U}]\cdot\nabla_{\mathbf{U}_{x_j}}\mathscr{H}_0[\mathbf{U}]\right). \tag{4}$$

Likewise, invariance by time translation implies that (3) contains the conservation law

$$\partial_t\mathscr{H}_0[\mathbf{U}] = \sum_j \partial_j\left(\nabla_{\mathbf{U}_{x_j}}\mathscr{H}_0[\mathbf{U}]\cdot\mathbf{J}\delta\mathscr{H}_0[\mathbf{U}]\right). \tag{5}$$

Finally, invariance by spatial translation implies that from (3) stems

$$\partial_t\left(\mathbb{Q}[\mathbf{U}]\right) = \nabla_{\mathbf{x}}\left(\frac{1}{2}\mathbf{J}\mathbf{U}\cdot\mathbf{J}\delta\mathscr{H}_0[\mathbf{U}] - \mathscr{H}_0[\mathbf{U}]\right) + \sum_\ell \partial_\ell(\mathbf{J}\delta\,\mathbb{Q}[\mathbf{U}]\cdot\nabla_{\mathbf{U}_{x_\ell}}\mathscr{H}_0[\mathbf{U}])\,. \tag{6}$$

The reader is referred to Appendix A for a derivation of the latter.

We are interested in the analysis of the dynamics of near-plane periodic uniformly traveling waves of (1). Let us first recall that a (uniformly traveling) wave is a solution whose time evolution occurs through the action of symmetries. We say that the wave is a plane wave when in a suitable frame it is constant in all but one direction, and that it is periodic if it is periodic up to symmetries. Given the foregoing set of symmetries, after choosing for the sake of concreteness, the direction of propagation as[2] $\mathbf{e}_1$ and the normalizing period

2. Throughout the text, we denote as $\mathbf{e}_j$ the jth vector of the canonical basis of $\mathbf{R}^d$. In particular, $\mathbf{e}_1 = (1,0,\dots,0)$.

to be 1 through the introduction of wavenumbers, we are interested in solutions to (1) of the form

$$\begin{aligned} f(t,\mathbf{x}) &= \mathrm{e}^{-\,\mathrm{i}\left(\underline{k}_\phi\,(x-\underline{c}_x\,t)+\underline{\omega}_\phi\,t\right)}\,\underline{f}(\underline{k}_x\,(x-\underline{c}_x\,t))\\ &= \mathrm{e}^{-\,\mathrm{i}(\underline{k}_\phi\,x+(\underline{\omega}_\phi-\underline{k}_\phi\,\underline{c}_x)\,t)}\,\underline{f}(\underline{k}_x\,x+\underline{\omega}_x\,t)\,, \end{aligned}$$

with profile $\underline{f}$ 1-periodic, wavenumbers $(\underline{k}_\phi,\underline{k}_x)\in\mathbf{R}^2$, $\underline{k}_x>0$, time frequencies $(\underline{\omega}_\phi,\underline{\omega}_x)\in\mathbf{R}^2$, spatial speed $\underline{c}_x\in\mathbf{R}$, where

$$\mathbf{x}=(x,\mathbf{y})\qquad \underline{\omega}_x=-\underline{k}_x\,\underline{c}_x\,.$$

In other terms, we consider solutions to (3) in the form

$$\mathbf{U}(t,\mathbf{x})=\mathrm{e}^{\left(\underline{k}_\phi\,(x-\underline{c}_x\,t)+\underline{\omega}_\phi\,t\right)\mathbf{J}}\,\underline{\mathcal{U}}(\underline{k}_x\,(x-\underline{c}_x\,t))\,, \tag{7}$$

with $\underline{\mathcal{U}}$ 1-periodic (and nonconstant). More general periodic plane waves are also considered in Appendix D. Beyond references to results involved in our analysis given along the text and comparison to the literature provided near each main statement, in order to place our contribution in a bigger picture, we refer the reader to [37] for a general background on nonlinear wave dynamics and to [46, 30, 14] for material more specific to Hamiltonian systems.

To set the frame for linearization, we observe that going to a frame adapted to the background wave in (7) by

$$\mathbf{U}(t,\mathbf{x})=\mathrm{e}^{\left(\underline{k}_\phi\,(x-\underline{c}_x\,t)+\underline{\omega}_\phi\,t\right)\mathbf{J}}\,\mathbf{V}(t,\underline{k}_x\,(x-\underline{c}_x\,t),\mathbf{y})\,,$$

changes (3) into

(8)

$$\begin{aligned} \partial_t\mathbf{V}&=\mathbf{J}\delta\mathscr{H}[\mathbf{V}]\,,\\ \mathscr{H}[\mathbf{V}]&:=\mathscr{H}_0(\mathbf{V},(\underline{k}_x\partial_x+\underline{k}_\phi\mathbf{J})\mathbf{V},\nabla_{\mathbf{y}}\mathbf{V})-\underline{\omega}_\phi\mathscr{M}[\mathbf{V}]+\underline{c}_x\mathbb{Q}_1(\mathbf{V},(\underline{k}_x\partial_x+\underline{k}_\phi\mathbf{J})\mathbf{V})\\ &=\mathscr{H}_0(\mathbf{V},(\underline{k}_x\partial_x+\underline{k}_\phi\mathbf{J})\mathbf{V},\nabla_{\mathbf{y}}\mathbf{V})-\left(\underline{\omega}_\phi-\underline{k}_\phi\,\underline{c}_x\right)\mathscr{M}[\mathbf{V}]-\underline{\omega}_x\mathbb{Q}_1[\mathbf{V}]\,, \end{aligned}$$

and that $(t,x,\mathbf{y})\mapsto\underline{\mathcal{U}}(x)$ is a stationary solution to (8). Direct linearization of (8) near this solution provides the linear equation $\partial_t\mathbf{V}=\mathcal{L}\,\mathbf{V}$ with $\mathcal{L}$ defined by

$$\mathcal{L}\mathbf{V}=\mathbf{J}\,\mathrm{Hess}\,\mathscr{H}[\underline{\mathcal{U}}](\mathbf{V})\,, \tag{9}$$

where Hess denotes the variational Hessian, that is, Hess $=L\delta$ with L denoting linearization. Incidentally, we point out that the natural splitting

$$\mathscr{H}_0=\mathscr{H}_0^x+\mathscr{H}^{\mathbf{y}}\,,\qquad \mathscr{H}^{\mathbf{y}}[\mathbf{U}]=\tfrac{1}{2}\kappa(\|\mathbf{U}\|^2)\|\nabla_{\mathbf{y}}\mathbf{U}\|^2\,,$$

may be followed all the way through frame change and linearization

$$\begin{aligned} \mathscr{H}&=\mathscr{H}^x+\mathscr{H}^{\mathbf{y}}\,,\\ \mathcal{L}&=\mathbf{J}\,\mathrm{Hess}\,\mathscr{H}^x[\underline{\mathcal{U}}]+\mathbf{J}\,\mathrm{Hess}\,\mathscr{H}^{\mathbf{y}}[\underline{\mathcal{U}}]=:\mathcal{L}^x+\mathcal{L}^{\mathbf{y}}\,, \end{aligned}$$

with $\mathcal{L}^{\mathbf{y}}=-\kappa(\|\underline{\mathcal{U}}\|^2)\mathbf{J}\,\Delta_{\mathbf{y}}$.

As made explicit in Section 3.1 at the spectral and linear level, to make the most of the spatial structure of periodic plane waves, it is convenient to introduce a suitable Bloch–Fourier integral transform. As a result, one may analyze the action of $\mathcal{L}$ defined on $L^2(\mathbf{R})$ through[3] the actions of $\mathcal{L}_{\xi,\boldsymbol{\eta}}$ defined on $L^2((0,1))$ with periodic boundary conditions, where $(\xi,\boldsymbol{\eta}) \in [-\pi,\pi]\times\mathbf{R}^{d-1}$, ξ being a longitudinal Floquet exponent and $\boldsymbol{\eta}$ a transverse Fourier frequency. The operator $\mathcal{L}_{\xi,\boldsymbol{\eta}}$ encodes the action of $\mathbf{J}\,\mathrm{Hess}\,\mathscr{H}[\underline{\mathcal{U}}]$ on functions of the form

$$\mathbf{x} = (x,\mathbf{y}) \mapsto \mathrm{e}^{\mathrm{i}\,\xi x + \mathrm{i}\,\boldsymbol{\eta}\cdot\mathbf{y}}\,\mathbf{W}(x)\,, \qquad \mathbf{W}(\cdot+1) = \mathbf{W}\,,$$

through

$$\mathbf{J}\,\mathrm{Hess}\,\mathscr{H}[\underline{\mathcal{U}}]\left((x,\mathbf{y}) \mapsto \mathrm{e}^{\mathrm{i}\,\xi x + \mathrm{i}\,\boldsymbol{\eta}\cdot\mathbf{y}}\,\mathbf{W}(x)\right)(\mathbf{x}) = \mathrm{e}^{\mathrm{i}\,\xi x + \mathrm{i}\,\boldsymbol{\eta}\cdot\mathbf{y}}\,(\mathcal{L}_{\xi,\boldsymbol{\eta}}\mathbf{W})(x)\,.$$

In particular, the spectrum of $\mathcal{L}$ coincides with the union over $(\xi,\boldsymbol{\eta})$ of the spectra of $\mathcal{L}_{\xi,\boldsymbol{\eta}}$. In turn, as recalled in Section 3.3, generalizing the analysis of Gardner [25], the spectrum of each $\mathcal{L}_{\xi,\boldsymbol{\eta}}$ may be conveniently analyzed with the help of an Evans function $D_\xi(\,\cdot\,,\boldsymbol{\eta})$, an analytic function whose zeros agree in location and algebraic multiplicity[4] with the spectrum of $\mathcal{L}_{\xi,\boldsymbol{\eta}}$. A large part of our spectral analysis hinges on the derivation of an expansion of $D_\xi(\lambda,\boldsymbol{\eta})$ when $(\lambda,\xi,\boldsymbol{\eta})$ is small (Theorem 3.2).

As derived in Section 2, families of plane periodic profiles in a fixed direction – here taken to be $\mathbf{e}_1$ – form four-dimensional manifolds when identified up to rotational and spatial translations, parametrized by $(\mu_x, c_x, \omega_\phi, \mu_\phi)$, where (μ_x,μ_ϕ) are constants of integration of profile equations associated with conservation laws (4) and (6) (or, more precisely, its first component since we consider waves propagating along $\mathbf{e}_1$). The averages along wave profiles of quantities of interest are expressed in terms of an action integral $\Theta(\mu_x, c_x, \omega_\phi, \mu_\phi)$ and its derivatives. This action integral plays a prominent role in our analysis. A significant part of our analysis, indeed, aims at reducing properties of operators acting on infinite-dimensional spaces to properties of this finite-dimensional function.

After these preliminary observations, we here give a brief account of each of our main results and provide only later in the text more specialized comments around precise statements. Our main achievements are essentially twofold. On the one hand, we provide counterparts to the main upshots of [12, 13, 9, 10, 11, 50] – derived for one-dimensional Hamiltonian equations of Korteweg type – for one-dimensional Hamiltonian equations of Schrödinger type. On the other hand, we extend parts of this analysis to the present multidimensional framework.

3. As, by using Fourier transforms on constant-coefficient operators one reduces their action on functions over the whole space to finite-dimensional operators parametrized by Fourier frequencies.

4. Defined, for the spectrum, as the rank of the residue of the resolvent map.