

PURSUING STACKS
(À LA POURSUITE DES CHAMPS)
VOLUME I

Alexandre Grothendieck

Édité par Georges Maltsiniotis

Documents Mathématiques
série dirigée par Patrick Popescu-Pampu

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ISSN : 1629-4939

ISBN 978-2-85629-958-6

Directeur de la publication : Fabien DURAND

DOCUMENTS MATHÉMATIQUES 20

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Avis au lecteur

Contrairement à ce que son titre suggère, *Pursuing Stacks* (ou du moins la partie du projet que Grothendieck a réalisée et qui devait s'intituler *The Modelizing Story* ou *Histoire de Modèles*) n'est pas consacré à la poursuite des champs, qui n'occupe que les treize premières sections ainsi que partiellement les sections 15–21 et 27. De plus, il s'agit surtout de ∞ -champs *sur le point*, autrement dit, des ∞ -groupoïdes faibles, les seules réflexions sur de ∞ -champs sur un topos arbitraire, comme coefficients naturels pour une cohomologie non abélienne, étant purement heuristiques. Le reste des cent quarante sections traite de la théorie de l'homotopie : recherche de modèles pour les types d'homotopie (et plus particulièrement de petites catégories dont la catégorie des préfaisceaux modélise canoniquement les types d'homotopie : les *catégories test*), structures homotopiques, structures de contractibilité et d'asphéricité, abélianisation et schématisation des types d'homotopie. Grothendieck pensait revenir ultérieurement sur la question des ∞ -champs sur un topos et développer, dans un ou deux volumes supplémentaires, ce qu'il avait esquissé dans ses lettres à Breen (lettres qu'il a intégrées dans *Pursuing Stacks* comme appendice), mais il ne l'a jamais fait. Néanmoins, la recherche des modèles pour les types d'homotopie n'est pas sans rapport avec les ∞ -champs, puisque d'après « l'hypothèse d'homotopie », conjecture fondamentale de Grothendieck, les ∞ -groupoïdes faibles doivent modéliser les types d'homotopie.

Le premier volume de cette édition comporte les quatre premiers chapitres, correspondant aux sections 1–91 et 95–98. Dans un second volume, on publiera les trois derniers chapitres, sections 92–94 et 99–140, les lettres à Breen, ainsi que la correspondance de Grothendieck avec de nombreux mathématiciens, autour des thèmes de *Pursuing Stacks*.

Notice to the reader

Despite what its title suggests, *Pursuing Stacks* (or at least the part of the project that Grothendieck carried out under the name of *The Modelizing Story* or *Histoire de Modèles*) is not about the pursuit of stacks. Only the thirteen first sections, as well as, partially, sections 15–21 and 27, are about stacks. Furthermore, it is mainly about ∞ -stacks *on the point*, i.e. weak ∞ -groupoids. The only reflections on stacks on arbitrary topoi, as natural coefficients for a non-abelian cohomology, are purely heuristic. The rest of the hundred and forty sections deals with homotopy theory : the search for models for homotopy types (and more particularly for small categories whose presheaf category models canonically homotopy types : the *test categories*), homotopy structures, contractibility and asphericity structures, abelianization and schematization of homotopy types. Grothendieck was planning to come back later to ∞ -stacks on topoi and to develop, in one or two additional volumes, what he had sketched out in his letters to Breen (letters that he included in *Pursuing Stacks* as an appendix), but he never did it. Nevertheless, the search for models for homotopy types is closely related to ∞ -stacks, since according to the “homotopy hypothesis”, a fundamental conjecture of Grothendieck, the weak ∞ -groupoids model homotopy types.

The first volume of this edition consists of the first four chapters (sections 1–91 and 95–98). In a second volume, we will publish the last three chapters, the letters to Breen, as well as the correspondence of Grothendieck with several mathematicians, around the themes of *Pursuing Stacks*.

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