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SINGULAR VECTORS AND GEOMETRY AT INFINITY OF PRODUCTS OF HYPERBOLIC SPACES

by Toshiaki Hattori

Abstract. — Let $\mathbf{k}$ be a number field and $\mathbf{k}_M$ the Minkowski space associated to $\mathbf{k}$. Dirichlet's theorem in Diophantine approximation is generalized to the case of approximations of vectors in $\mathbf{k}_M$ by elements of $\mathbf{k}$. We study the set of singular elements of $\mathbf{k}_M$ in this setting and calculate its Hausdorff dimension, by relating the inequalities to Tits geometry of the geometric boundary of the symmetric space naturally associated to $\mathbf{k}$.

Résumé (*Vecteurs singuliers et géométrie à l'infini des produits d'espaces hyperboliques*). — Soient $\mathbf{k}$ un corps de nombres algébriques et $\mathbf{k}_M$ l'espace de Minkowski associé à $\mathbf{k}$. Le théorème de Dirichlet en approximation diophantienne se généralise au cas de l'approximation de vecteurs dans $\mathbf{k}_M$ par des éléments de $\mathbf{k}$. Nous étudions l'ensemble des éléments singuliers de $\mathbf{k}_M$ dans ce cadre et nous calculons sa dimension de Hausdorff, en reliant les inégalités définissant les vecteurs singuliers à la géométrie de Tits du bord géométrique de l'espace symétrique naturellement associé à $\mathbf{k}$.

1. Introduction

In the classical theory of Diophantine approximation, a real number x is said to be badly approximable if there exists a positive constant C such that

$$|q|\,|qx - p| \geq C$$

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for all integers p, q with $q \neq 0$; a real number y is called singular if, for every $\varepsilon > 0$, there exists a positive constant $C_0(\varepsilon)$ such that the set of inequalities

$$0 < q \leq C, \quad |qy - p| < \varepsilon C^{-1}$$

has an integral solution (p, q) for all C greater than $C_0(\varepsilon)$. The set of badly approximable numbers has zero Lebesgue measure and is thick ([4], [26]). By contrast, the set of singular numbers is identical to the set of rational numbers, and hence its Hausdorff dimension is equal to zero (see [7, p. 94]).

The goal of this paper is to generalize the latter result into the setting of algebraic number fields. More precisely, let $\mathbf{k}$ be a number field of degree $d = l + 2m$ with l real places and m complex places. Let $\iota_1, \ldots, \iota_l : \mathbf{k} \longrightarrow \mathbf{R}$ be the real embeddings if l is positive, and let $\iota_{l+1}, \ldots, \iota_{l+m} : \mathbf{k} \longrightarrow \mathbf{C}$ be the complex embeddings that are not complex conjugate to each other if m is positive. We approximate elements of the Minkowski space $\mathbf{k}_M = \mathbf{R}^l \times \mathbf{C}^m$ associated to $\mathbf{k}$ by elements of $\mathbf{k}$ through the twisted diagonal embedding $\iota_{\mathbf{k}} : \mathbf{k} \longrightarrow \mathbf{k}_M$ given by

$$\iota_{\mathbf{k}}(a) = (\iota_1(a), \ldots, \iota_{l+m}(a)) \quad \text{for } a \in \mathbf{k} \, .$$

For $\xi = (\xi_1, \ldots, \xi_{l+m}), \mu = (\mu_1, \ldots, \mu_{l+m}) \in \mathbf{k}_M$, we define their sum and product by

$$\xi + \mu = (\xi_1 + \mu_1, \, \ldots, \, \xi_{l+m} + \mu_{l+m}) \, , \quad \xi \cdot \mu = (\xi_1 \mu_1, \, \ldots, \, \xi_{l+m} \mu_{l+m})$$

and we equip $\mathbf{k}_M$ with the sup norm

$$\|\xi\| = \max_{1 \leq i \leq l+m} |\xi_i| \, ,$$

where $|\cdot|$ is the usual Euclidean absolute value on $\mathbf{R}$ or $\mathbf{C}$. Let $\mathcal{O}_{\mathbf{k}}$ be the ring of integers of $\mathbf{k}$.

The following generalization of Dirichlet's theorem (cf. [32, Chapter I]) is obtained from a result of R. Quême ([30]).

Theorem 1.1 (cf. [30]). — *There exists a positive constant C depending only on $\mathbf{k}$ such that for every $\xi \in \mathbf{k}_M - \iota_{\mathbf{k}}(\mathbf{k})$, there are infinitely many $\beta = p/q\, ;\, p \in \mathcal{O}_{\mathbf{k}}$, $q \in \mathcal{O}_{\mathbf{k}} - \{0\}$ satisfying*

$$\|\iota_{\mathbf{k}}(q) \cdot \xi - \iota_{\mathbf{k}}(p)\| < C \|\iota_{\mathbf{k}}(q)\|^{-1} \, . \tag{1}$$

A vector $\xi \in \mathbf{k}_M$ is called $\mathbf{k}$-badly approximable if there exists a positive constant C depending on ξ such that

$$\|\iota_{\mathbf{k}}(q)\| \, \|\iota_{\mathbf{k}}(q) \cdot \xi - \iota_{\mathbf{k}}(p)\| \geq C \tag{2}$$

for any $p \in \mathcal{O}_{\mathbf{k}}$ and $q \in \mathcal{O}_{\mathbf{k}} - \{0\}$ (see [13]). Let $\mathrm{Bad}(\mathbf{k})$ be the set of $\mathbf{k}$-badly approximable vectors in $\mathbf{k}_M$.

In this paper, we say that $\xi \in \mathbf{k}_M$ is a $\mathbf{k}$-singular vector if, for each $\varepsilon > 0$, the set of inequalities

$$\|\iota_{\mathbf{k}}(q)\| \leq C, \quad \|\iota_{\mathbf{k}}(q) \cdot \xi - \iota_{\mathbf{k}}(p)\| < \varepsilon C^{-1}$$

has a solution $(p, q) \in (\mathcal{O}_{\mathbf{k}})^2$ with $q \neq \mathbf{0}$ for all C greater than some positive constant $C_0(\varepsilon)$ depending on ξ and ε. Otherwise, we say that ξ is $\mathbf{k}$-regular. Note that $\mathbf{k}$-badly approximable vectors are $\mathbf{k}$-regular. Let $\mathrm{Sing}(\mathbf{k})$ be the set of $\mathbf{k}$-singular vectors.

The set $\mathrm{Bad}(\mathbf{k})$ has already been studied in [4], [10], [13], [15] and [27]. Generalizing the results in the case of the rational field and the results [10, Theorem 5.2], [15] in the case of imaginary quadratic fields with class number 1, M. Einsiedler, A. Ghosh, and B. Lytle showed the following.

Theorem 1.2 ([13]). — *The set* $\mathrm{Bad}(\mathbf{k})$ *has zero Lebesgue measure in* $\mathbf{k}_M$ *when we regard* $\mathbf{k}_M$ *as* $\mathbf{R}^d$. *It is also thick, and its Hausdorff dimension* $\dim_{\mathrm{H}}(\mathrm{Bad}(\mathbf{k}))$ *is equal to* d.

To state our results on $\mathrm{Sing}(\mathbf{k})$ we introduce an integral-valued function $f_{\mathbf{k}}$ on $\mathbf{k}_M$. For any finite set S, let $\#S$ denote the cardinality of S. For any nonnegative integer q, let $\mathbf{N}(q)$ be the set of all positive integers smaller than $q+1$. We also write $a^{(j)}$ instead of $\iota_j(a)$ for $a \in \mathbf{k}$. Let $\xi = (\xi_1, \ldots, \xi_{l+m}) \in \mathbf{k}_M$. We define a subset $A(\xi)$ of $\{0, \ldots, l\} \times \{0, \ldots, m\}$ as follows: $(\lambda, \mu) \in A(\xi)$ if and only if there exist a subset I_1 of $\mathbf{N}(l)$, a subset I_2 of $\mathbf{N}(l+m) \smallsetminus \mathbf{N}(l)$, and an element η of $\mathbf{k}$ such that $\#I_1 = \lambda$, $\#I_2 = \mu$, and

$$\xi_k \neq \eta^{(k)} \text{ for } k \in I_1 \cup I_2 \,, \quad \xi_k = \eta^{(k)} \text{ for } k \notin I_1 \cup I_2 \,.$$

We define a function $f_{\mathbf{k}} : \mathbf{k}_M \longrightarrow \mathbf{Z}$ by

$$f_{\mathbf{k}}(\xi) = \min\{\lambda + 2\mu \,|\, (\lambda,\, \mu) \in A(\xi)\} \quad \text{for } \xi \in \mathbf{k}_M \,. \tag{3}$$

We remark that $\xi \in \iota_{\mathbf{k}}(\mathbf{k})$ if and only if $f_{\mathbf{k}}(\xi) = 0$.

Generalizing the result in the case $\mathbf{k} = \mathbf{Q}$, we show the following.

Theorem 1.3. — *Let* $\xi \in \mathbf{k}_M = \mathbf{R}^l \times \mathbf{C}^m$. *Then* ξ *is* $\mathbf{k}$*-singular if and only if* $f_{\mathbf{k}}(\xi) < d/2$.

Theorem 1.4. — *Let*

$$d' = \begin{cases} (d-1)/2 & \textit{if } d \textit{ is odd} \\ d/2 - 2 & \textit{if } d \textit{ is even, } l = 0 \textit{ and } m \textit{ is even} \\ d/2 - 1 & \textit{otherwise.} \end{cases}$$

Then the Hausdorff dimension of the subset $\mathrm{Sing}(\mathbf{k})$ *of* $\mathbf{k}_M$ *is equal to* d' *when we regard* $\mathbf{k}_M$ *as* $\mathbf{R}^d$. *The set* $\mathrm{Sing}(\mathbf{k})$ *is identical to* $\iota_{\mathbf{k}}(\mathbf{k})$ *if and only if* $\mathbf{k}$ *is the rational field* $\mathbf{Q}$ *or a quadratic field or a totally complex quartic field.*

Theorem 1.3 also provides a criterion of being "not $\mathbf{k}$-badly approximable" in the case $\mathrm{Sing}(\mathbf{k}) \neq \iota_{\mathbf{k}}(\mathbf{k})$. Although measure theoretic properties of $\mathrm{Bad}(\mathbf{k})$ are well studied, it is another problem to know whether or not a given element of $\mathbf{k}_M$ is $\mathbf{k}$-badly approximable.

In the classical case, it is well known that a real number is badly approximable if and only if the partial quotients in its continued fraction expansion are bounded (cf. Theorem 5F of [32, Chapter I]). This was generalized to the case of $\mathbf{Q}(\sqrt{-3})$ in [9], the case of arbitrary imaginary quadratic field in [25]. E. Burger and R. Hines found methods to construct $\mathbf{k}$-badly approximable vectors for any $\mathbf{k}$ different from $\mathbf{Q}$ ([6], [24]). There is another method to construct $\mathbf{k}$-badly approximable vectors in the case $l+m \geq 2$: Proposition 3.1 of [13] and the argument in the proof of Proposition 8.5 of [20] show that $(b, \ldots, b) \in \mathbf{k}_M$ is $\mathbf{k}$-badly approximable for any badly approximable real number b.

On the other hand, there were no known concrete criteria to ensure that a vector in $\mathbf{k}_M$ is not $\mathbf{k}$-badly approximable in the case $l + m \geq 2$, even though $\mathrm{Bad}(\mathbf{k})$ is a set of measure zero.

Example. — Let ζ be a primitive complex 7th root of unity and $\mathbf{k}$ the cyclotomic field $\mathbf{Q}(\zeta)$. Let $a \in \mathbf{k}$ and b be a badly approximable real number that is not contained in $\mathbf{k}$. Then $(a^{(1)}, a^{(2)}, b), (a^{(1)}, b, a^{(3)}), (b, a^{(2)}, a^{(3)}) \in \mathbf{k}_M = \mathbf{C}^3$ are not $\mathbf{k}$-badly approximable, while $(b, b, b) \in \mathbf{k}_M$ is $\mathbf{k}$-badly approximable.

We outline the proofs of the main results. Our approach is based on Riemannian geometry of nonpositively curved manifolds (see [1], [11] and [12]).

In [17], L.R. Ford found that Dirichlet's theorem is closely related to the geometry of the hyperbolic plane $\mathbf{H}$. He considered horoballs in the upper half-plane $\mathbf{H}$ together with the action of $SL(2, \mathbf{Z})$ on $\mathbf{H}$. After Ford, connections between Diophantine approximation problems and the geometry of hyperbolic spaces, or more generally Gromov hyperbolic spaces, were studied extensively by many authors (see, for example, [23], [16] and references therein). Some product spaces of hyperbolic spaces were also used in [20] and [28] for approximation by algebraic numbers.

From this point of view, it is natural in our case to consider horoballs in the product of l copies of the hyperbolic plane and m copies of the 3-dimensional hyperbolic space together with the action of $\mathit{\Gamma} = SL(2, \mathcal{O}_{\mathbf{k}})$ on this product.

Let $V = SL(2, \mathbf{R})/SO(2)$ and $\widehat{V} = SL(2, \mathbf{C})/SU(2)$. We identify V (respectively $\widehat{V}$) with the upper half-plane $\mathbf{H}$ (respectively the three-dimensional upper half-space $\mathcal{H}$) in the usual manner (see Section 2 for a more precise description), and equip V (respectively $\widehat{V}$) with the Poincaré metric (respectively the metric that is twice the Poincaré metric on $\mathcal{H}$). The group $G = SL(2, \mathbf{R})^l \times SL(2, \mathbf{C})^m$ acts on the Riemannian product $\widetilde{V} = V^l \times \widehat{V}^m$ by

$$g \cdot z = (g_1 \cdot x_1, \ldots, g_l \cdot x_l, g_{l+1} \cdot \widehat{x}_{l+1}, \ldots, g_{l+m} \cdot \widehat{x}_{l+m}) \tag{4}$$

for $z = (x_1, \dots, x_l, \widehat{x}_{l+1}, \dots, \widehat{x}_{l+m}) \in \widetilde{V}$ and $g = (g_1, \dots, g_{l+m}) \in G$, where $x_1, \dots, x_l \in V$ and $\widehat{x}_{l+1}, \dots, \widehat{x}_{l+m} \in \widehat{V}$. Note that, in the case $l = 0$ or $m = 0$, an obvious modification should be made to this formula (4). However, in the rest of this paper, we omit to write such modified formulae in order to avoid complicating the description. We extend each embedding ι_j to an embedding of $SL(2, \mathbf{k})$ into $SL(2, \mathbf{R})$ or $SL(2, \mathbf{C})$ by

$$\iota_j(g) = \begin{pmatrix} \iota_j(p) & \iota_j(r) \\ \iota_j(q) & \iota_j(s) \end{pmatrix} \quad \text{for } g = \begin{pmatrix} p & r \\ q & s \end{pmatrix} \in SL(2, \mathbf{k}) \ . \tag{5}$$

Then the twisted diagonal embedding $\iota_{\mathbf{k}}$ can be extended to an embedding $SL(2, \mathbf{k}) \longrightarrow G$ by

$$\iota_{\mathbf{k}}(g) = (\iota_1(g), \dots, \iota_{l+m}(g)) \quad \text{for } g \in SL(2, \mathbf{k}) \ . \tag{6}$$

The group $SL(2, \mathbf{k})$ acts isometrically on $\widetilde{V}$ through this embedding:

$$\iota_{\mathbf{k}}(g) \cdot z = (\iota_1(g) \cdot x_1, \ \dots \ , \iota_l(g) \cdot x_l, \ \iota_{l+1}(g) \cdot \widehat{x}_{l+1}, \ \dots, \ \iota_{l+m}(g) \cdot \widehat{x}_{l+m}) \tag{7}$$

for $z = (x_1, \dots, x_l, \widehat{x}_{l+1}, \dots, \widehat{x}_{l+m}) \in \widetilde{V}$ and $g \in SL(2, \mathbf{k})$.

Let $x_0 \in V$ be the coset of the identity element of $SL(2, \mathbf{R})$, $\widehat{x}_0 \in \widehat{V}$ the coset of the identity element of $SL(2, \mathbf{C})$, and let

$$z_0 = (x_0, \dots, x_0, \widehat{x}_0, \dots, \widehat{x}_0) \in \widetilde{V} \ . \tag{8}$$

Then the isotropy subgroup of G at z_0 is $K = SO(2)^l \times SU(2)^m$, and $\widetilde{V}$ is diffeomorphic to the quotient space G/K.

We associate each element of $\mathbf{k}_M$ with a certain geodesic ray of $\widetilde{V}$ as follows. Let $\xi = (\xi_1, \dots, \xi_{l+m}) \in \mathbf{k}_M$. We define an element u_ξ of G by

$$u_\xi = \left(\begin{pmatrix} 1 & \xi_1 \\ 0 & 1 \end{pmatrix}, \dots, \begin{pmatrix} 1 & \xi_{l+m} \\ 0 & 1 \end{pmatrix} \right) \tag{9}$$

and define a geodesic ray $\gamma_\xi : [0, \infty) \longrightarrow \widetilde{V}$ by

$$\gamma_\xi(t) = u_\xi g_t \cdot z_0 \quad \text{for } t \geq 0 \ , \tag{10}$$

where

$$g_t = \left(\begin{pmatrix} e^{-t/\sqrt{4d}} & \\ & e^{t/\sqrt{4d}} \end{pmatrix}, \dots, \begin{pmatrix} e^{-t/\sqrt{4d}} & \\ & e^{t/\sqrt{4d}} \end{pmatrix} \right) \in G \ . \tag{11}$$

Let $\Pi : \widetilde{V} \longrightarrow \iota_{\mathbf{k}}(\Gamma) \backslash \widetilde{V}$ be the projection to the noncompact quotient space. For any geodesic ray γ of $\widetilde{V}$, we say that $\Pi \circ \gamma$ is divergent if, for any given compact subset W of $\iota_{\mathbf{k}}(\Gamma) \backslash \widetilde{V}$, there exists $t_0 \geq 0$ such that $\Pi \circ \gamma(t) \notin W$ for $t \geq t_0$. This condition is a paraphrase of another one written in terms of horoballs in $\widetilde{V}$ and the action of $\iota_{\mathbf{k}}(\Gamma)$ on $\widetilde{V}$ (see Proposition 3.2).

Proposition 1.5. — *$\Pi \circ \gamma_\xi$ is divergent if and only if ξ is a $\mathbf{k}$-singular vector.*