Bull. Soc. Math. France
150 (1), 2022, p. 1-16

MAXIMAL ESTIMATES FOR THE KRAMERS–FOKKER–PLANCK OPERATOR WITH ELECTROMAGNETIC FIELD

by Bernard Helffer & Zeinab Karaki

Abstract. — In continuation of a former work by the first author with F. Nier (2009) and of a more recent work by the second author on the torus (2019), we consider the Kramers–Fokker–Planck operator (KFP) with an external electromagnetic field on $\mathbb{R}^d$. We show a maximal type estimate on this operator using a nilpotent approach for vector field polynomial operators and induced representations of a nilpotent graded Lie algebra. This estimate leads to an optimal characterization of the domain of the closure of the (KFP) operator and a criterion for the compactness of the resolvent.

Résumé (*Estimation maximale pour l'opérateur de Kramers-Fokker-Planck avec champ électromagnétique*). — Dans la continuité d'un travail antérieur du premier auteur avec F. Nier (2009) et d'un travail plus récent du deuxième auteur sur le tore (2019), nous considérons l'opérateur de Kramers-Fokker-Planck (KFP) avec un champ électromagnétique sur $\mathbb{R}^d$. Nous montrons une estimation de type maximal sur cet opérateur en utilisant une approche nilpotente pour les opérateurs polynômes de champs de vecteurs et des représentations induites d'une algèbre de Lie graduée nilpotente. Cette estimation conduit à une caractérisation optimale du domaine de la fermeture de l'opérateur (KFP) et à un critère de compacité de la résolvante.

Texte reçu le 25 janvier 2021, accepté le 20 juillet 2021.

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Mathematical subject classification (2010). — 35H10, 47A67; 82C40, 82D40.

Key words and phrases. — Fokker–Planck equation, magnetic field, electric potential, Lie algebra, irreducible representation, maximal estimate.

doi:10.24033/bsmf.2842

1. Introduction and main results

1.1. Introduction. — The Fokker–Planck equation was introduced by Fokker and Planck at the beginning of the twentieth century to describe the evolution of the density of particles under Brownian motion. In recent years, global hypoelliptic estimates have led to new results motivated by applications to the kinetic theory of gases. In this direction, many authors have shown maximal estimates to deduce the compactness of the resolvent of the Fokker–Planck operator and to have resolvent estimates in order to address the issue of return to the equilibrium. F. Hérau and F. Nier in [5] highlighted the links between the Fokker–Planck operator with a confining potential and the associated Witten Laplacian. Later, this work was extended in the book of B. Helffer and F. Nier [2], and we refer more specifically to their Chapter 9 for a proof of the maximal estimate.

In this article, we continue the study of the model case of the operator of Fokker–Planck with an external magnetic field B_e, which was initiated in the case of the torus $\mathbb{T}^d$ ($d = 2, 3$) in [9, 10], by considering $\mathbb{R}^d$ and reintroducing an electric potential as in [2]. In this context, we establish a maximal-type estimate for this model, giving a characterization of the domain of its closed extension and giving sufficient conditions for the compactness of the resolvent.

1.2. Statement of the result. — For $d = 2$ or 3, we consider, for a given external electromagnetic field B_e defined on $\mathbb{R}^d$ with value in $\mathbb{R}^{d(d-1)/2}$ and a real valued electric potential V defined on $\mathbb{R}^d$, the associated Kramers–Fokker–Planck operator K (in short KFP) defined by:

$$K = v \cdot \nabla_x - \nabla_x V \cdot \nabla_v - (v \wedge B_e) \cdot \nabla_v - \Delta_v + v^2/4 - d/2 \,, \tag{1}$$

where $v \in \mathbb{R}^d$ represents the velocity, $x \in \mathbb{R}^d$ represents the space variable, and the notation $(v \wedge B_e) \cdot \nabla_v$ means:

$$(v \wedge B_e) \cdot \nabla_v = \begin{cases} b(x)\,(v_1\partial_{v_2} - v_2\partial_{v_1}) & \text{if } d = 2 \\ b_1(x)(v_2\partial_{v_3} - v_3\partial_{v_2}) + b_2(x)(v_3\partial_{v_1} - v_1\partial_{v_3}) & \\ \qquad + b_3(x)(v_1\partial_{v_2} - v_2\partial_{v_1}) & \text{if } d = 3\,. \end{cases}$$

The operator K is initially considered as an unbounded operator on the Hilbert space $L^2(\mathbb{R}^d \times \mathbb{R}^d)$, whose domain is $D(K) = C_0^\infty(\mathbb{R}^d \times \mathbb{R}^d)$.

We then denote by:

- $K_{\min}$ the minimal extension of K where $D(K_{min})$ is the closure of $D(K)$ with respect to the graph norm;
- $K_{\max}$ the maximal extension of K where $D(K_{\max})$ is given by:

$$D(K_{\max}) = \{u \in L^2(\mathbb{R}^d \times \mathbb{R}^d) \mid Ku \in L^2(\mathbb{T}^d \times \mathbb{R}^d)\}\,.$$

We will use the notation $\mathbf{K}$ for the operator K_{min} or $\mathbf{K}_{B_e,V}$ if we want to mention the reference to B_e and V.

The existence of a strongly continuous semigroup associated to operator $\mathbf{K}$ when the magnetic field is regular, and $V = 0$ is shown in [9]. We will improve this result by considering a much lower regularity. In order to obtain the maximal accretivity, we are led to substitute the hypoellipticity argument by a regularity argument for the operators with coefficients in L^∞_{loc}, which will be combined with the more classical results of Rothschild–Stein in [12] for Hörmander operators of type 2 (see [6] for more details of this subject). Our first result is:

Theorem 1.1. — *If $B_e \in L^\infty_{loc}(\mathbb{R}^d, \mathbb{R}^{d(d-1)/2})$ and $V \in W^{1,\infty}_{loc}(\mathbb{R}^d)$, then $\mathbf{K}_{B_e,V}$ is maximally accretive.*

The theorem implies that the domain of the operator $\mathbf{K} = K_{min}$ has the following property:

$$D(\mathbf{K}) = D(K_{\max}) \,. \tag{2}$$

We are next interested in specifying the domain of the operator $\mathbf{K}$ introduced in (2). For this goal, we will establish a maximal estimate for $\mathbf{K}$, using techniques that were initially developed for the study of hypoellipticity of invariant operators on nilpotent groups and the proof of the Rockland conjecture. Before we state our main result, we introduce the following functional spaces:

- $B^2(\mathbb{R}^d)$ (or B^2_v to indicate the name of the variables) denotes the space:

$$B^2(\mathbb{R}^d) := \{u \in L^2(\mathbb{R}^d) \,|\, \forall(\alpha,\beta) \in \mathbb{N}^{2d},\ |\alpha| + |\beta| \le 2\,,\ v^\alpha\, \partial_v^\beta\, u \in L^2(\mathbb{R}^d)\}\,,$$

  which is equipped with its natural Hilbertian norm.
- $\tilde{B}^2(\mathbb{R}^d \times \mathbb{R}^d)$ is the space $L^2_x \widehat{\otimes} B^2_v$ (in $L^2(\mathbb{R}^d \times \mathbb{R}^d)$ identified with $L^2_x \widehat{\otimes} L^2_v$) with its natural Hilbert norm.
- $\mathcal{H}^2(\mathbb{R}^{2d})$ is the Sobolev space of degree 2 associated with the vector fields $\frac{\partial}{\partial v_j}$ $(j = 1, \ldots, d)$, $i\, v_\ell$ $(\ell = 1, \ldots, d)$ with weight 1 and $v \cdot \nabla_x$ with weight 2 as introduced in [10, Section 2]. It also reads

$$\mathcal{H}^2(\mathbb{R}^{2d}) = \{u \in \tilde{B}^2(\mathbb{R}^{2d})\,,\ v \cdot \nabla_x u \in L^2(\mathbb{R}^{2d})\}\,.$$

- $\mathcal{H}^2_{loc}(\mathbb{R}^{2d})$ is the space of functions that are locally in $\mathcal{H}^2(\mathbb{R}^{2d})$.

We can now state the second theorem of this article:

Theorem 1.2. — *Let $d = 2$ or 3. We assume that $B_e \in \mathcal{C}^1(\mathbb{R}^d, \mathbb{R}^{d(d-1)/2}) \cap L^\infty(\mathbb{R}^d, \mathbb{R}^{d(d-1)/2})$, $V \in C^2(\mathbb{R}^d, \mathbb{R})$ and that there exist positive constants C, $\rho_0 > \frac{1}{3}$, and $\gamma_0 < \frac{1}{3}$, such that*

$$|\nabla_x B_e(x)| \le C \langle \nabla V(x) \rangle^{\gamma_0}\,, \tag{3}$$

$$|D^\alpha_x V(x)| \le C \langle \nabla V(x) \rangle^{1-\rho_0}\,, \forall \alpha \ s.t.\ |\alpha| = 2\,, \tag{4}$$

where

$$\langle \nabla V(x) \rangle = \sqrt{|\nabla V(x)|^2 + 1}\,.$$

Then there exists $C_1 > 0$ such that, for all $u \in C_0^\infty(\mathbb{R}^d \times \mathbb{R}^d)$, the operator K satisfies the following maximal estimate:

$$\begin{aligned}(5)\qquad & \left\| |\nabla V(x)|^{\frac{2}{3}}\, u \right\| + \|(v \cdot \nabla_x - \nabla_x V \cdot \nabla_v - (v \wedge B_e) \cdot \nabla_v) u\| + \|u\|_{\tilde{B}^2} \\ & \qquad \leq C_1(\|Ku\| + \|u\|) \,.\end{aligned}$$

The proofs will combine the previous works of [2] (in the case $B_e = 0$) and [10] (in the case $V = 0$) with, in addition, two differences:

- $\mathbb{T}^d$ is replaced by $\mathbb{R}^d$.
- The reference operator in the enveloping algebra of the nilpotent algebra is different.

Notice also that when $B_e = 0$, our assumptions are weaker than in [2] where the property that $|\nabla V(x)|$ tends to $+\infty$ as $|x| \to +\infty$ was used to construct the partition of unity.

Using the density of $C_0^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ in the domain of $\mathbf{K}$, we obtain the following characterization of this domain:

Corollary 1.3. —

$$\begin{aligned}(6)\qquad D(\mathbf{K}) = \big\{u \in \tilde{B}^2(\mathbb{R}^d \times \mathbb{R}^d) \,\big|\, & (v \cdot \nabla_x - \nabla V \cdot \nabla_v - (v \wedge B_e) \cdot \nabla_v) u \\ & \textit{and } |\nabla V(x)|^{\frac{2}{3}}\, u \in L^2(\mathbb{R}^d \times \mathbb{R}^d)\big\}.\end{aligned}$$

In particular, this implies that under the assumptions of Theorem 1.2, the operator $\mathbf{K}_{B_e,V}$ has a compact resolvent if and only if $\mathbf{K}_{B_e=0,V}$ has the same property. This is, in particular, the case (see [2]) when

$$|\nabla V(x)| \to +\infty \text{ when } |x| \to +\infty \,,$$

as can also be seen directly from (6).

Remark 1.4. — For more results in the case without a magnetic field, we refer the reader to [2] and recent results obtained in 2018 by Wei-Xi Li [11] and in 2019 by M. Ben Said [1] in connection with a conjecture of Helffer and Nier relating the compact resolvent property for the (KFP)-operator with the same property for the Witten Laplacian on (0)-forms: $-\Delta_{x,v} + \frac{1}{4}|\nabla\Phi|^2 - \frac{1}{2}\Delta\Phi$, with $\Phi(x,v) = V(x) + \frac{v^2}{2}$. Its proof also involves nilpotent techniques. One can then naturally ask about results when $B_e(x)$ is unbounded. In particular, the existence of (KFP)-magnetic bottles (i.e., the compact resolvent property for the (KFP)-operator) when $V = 0$ is natural. Here, we simply observe that Proposition 5.19 in [2] holds (with exactly the same proof) for $\mathbf{K}_{B_e,V}$ when B_e and V are C^∞. Hence, there are no (KFP)-magnetic bottles.

2. Maximal accretivity for the Kramers–Fokker–Planck operator with a weakly regular electromagnetic field

To prove Theorem 1.1, we will show the Sobolev regularity associated with the following problem

$$K^* f = g \text{ with } f, g \in L^2_{loc}(\mathbb{R}^{2d}) \,,$$

where K^* is the formal adjoint of K:

$$K^* = -v\cdot\nabla_x - \Delta_v + (v\wedge B_e + \nabla_x V)\cdot\nabla_v + v^2/4 - d/2 \,. \tag{7}$$

The result of Sobolev regularity is the following:

Theorem 2.1. — *Let $d = 2$ or 3. We suppose that $B_e \in L^\infty_{loc}(\mathbb{R}^d, \mathbb{R}^{d(d-1)/2})$, and $V \in W^{1,\infty}_{loc}(\mathbb{R}^d\times\mathbb{R}^d)$. Then, for all $f \in L^2_{loc}(\mathbb{R}^{2d})$, such that $K^* f = g$ with $g \in L^2_{loc}(\mathbb{R}^{2d})$, $f \in \mathcal{H}^2_{loc}(\mathbb{R}^{2d})$.*

Before proving Theorem 2.1, we recall the following result :

Proposition 2.2 (Proposition A.3 in [10]). — *Let $c_j \in L^{\infty,2}_{loc}(\mathbb{R}^d\times\mathbb{R}^d)$, $j = 1,\dots,d$, where $L^{\infty,2}_{loc}(\mathbb{R}^d\times\mathbb{R}^d) = \{u \in L^2_{loc}, \forall\varphi \in C_0^\infty(\mathbb{R}^d\times\mathbb{R}^d)$ such that $\varphi u \in L^\infty_x(L^2_v)\}$, such that*

$$\partial_{v_j}\left(c_j(x,v)\right) = 0 \;\; \text{in } \mathcal{D}'(\mathbb{R}^{2d})\,, \quad \forall j = 1,\dots,d\,. \tag{8}$$

Let P_0 be the Kolmogorov operator

$$P_0 := -v\cdot\nabla_x - \Delta_v\,. \tag{9}$$

If $h \in L^2_{loc}(\mathbb{R}^{2d})$ satisfies

$$\begin{cases} P_0 h = \displaystyle\sum_{j=1}^{d} c_j(x,v)\,\partial_{v_j} h_j + \tilde{g} \\ h_j, \tilde{g} \in L^2_{loc}(\mathbb{R}^{2d}),\; \forall j = 1,\dots,d\,, \end{cases} \tag{10}$$

then $\nabla_v h \in L^2_{loc}(\mathbb{R}^{2d}, \mathbb{R}^d)$.

We can now give the proof of Theorem 2.1.

Proof of Theorem 2.1. — The proof is similar to that of Theorem A.2 in [10]. In the following, we will only focus on the differences appearing in our case. To show the Sobolev regularity of the problem $K^* f = g$ with f and $g \in L^2_{loc}(\mathbb{R}^{2d})$