Bull. Soc. Math. France
150 (1), 2022, p. 77-109

FINITENESS AND PERIODICITY OF CONTINUED FRACTIONS OVER QUADRATIC NUMBER FIELDS

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Abstract. — In this paper, we prove a periodicity theorem for certain continued fractions with partial quotients in the ring of integers of a fixed quadratic field. This theorem generalizes the classical theorem of Lagrange to a large set of continued fraction expansions.

As an application we consider the β-continued fractions and show that for any quadratic Perron number β, the β-continued fraction expansion of elements in $\mathbb{Q}(\beta)$ is either finite or eventually periodic.

More generally, we examine the finiteness and periodicity of the β-continued fractions for all quadratic integers β, thus studying problems raised by Rosen and Bernat.

Résumé (*Finitude et périodicité des fractions continues sur des corps de nombres quadratiques*). — Dans cet article, nous prouvons un théorème de périodicité pour certaines fractions continues avec les quotients incomplets dans l'anneau des entiers d'un corps quadratique fixé, qui généralise le théorème classique de Lagrange.

Texte reçu le 29 mars 2021, modifié le 2 novembre 2021, accepté le 29 novembre 2021.

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Mathematical subject classification (2010). — 11A55, 11J70, 11A63.

Key words and phrases. — Continued fraction, Perron number, quadratic Pisot numbers, finiteness.

This work was supported by the project CZ.02.1.01/0.0/0.0/16_019/0000778 of the Czech Technical University in Prague and projects PRIMUS/20/SCI/002 and UNCE/SCI/022 from Charles University. We thank the Centro di Ricerca Matematica Ennio De Giorgi for support. The third author is a member of the GNSAGA research group of INdAM..

doi:10.24033/bsmf.2845

Comme application, nous considérons les β-fractions continues et montrons que pour tout nombre de Perron quadratique β, le développement en β-fractions continues des éléments dans $\mathbb{Q}(\beta)$ est soit fini, soit éventuellement périodique.

Plus généralement, nous examinons la finitude et la périodicité des β-fractions continues pour tous les entiers quadratiques β, étudiant ainsi des problèmes soulevés par Rosen et Bernat.

1. Introduction

1.1. A periodicity result for continued fractions with partial quotients in a real quadratic field. — As is well known, a theorem due to Lagrange states that the simple continued fraction expansion of quadratic irrationals is periodic. In this paper, we present an extension of Lagrange's theorem, which applies to many different continued fraction expansions. Our main theorem is the following:

THEOREM 1.1. — *Let $\xi \in \mathbb{R}$ be a quadratic irrational number, let $\mathbb{Q}(\xi) = \mathbb{Q}(\sqrt{D})$ (where D a positive squarefree integer) be the field generated by ξ over $\mathbb{Q}$ and denote by $\mathcal{O}$ its ring of integers.*

Let $\xi = [a_0, a_1, \dots]$ be any infinite continued fraction converging to ξ and such that the following conditions are satisfied for every $n \geq 0$:

$$a_n \in \mathcal{O}, \qquad a_n \geq 1, \qquad |a_n'| \leq a_n,$$

where a_n' denotes the image of a_n under the nontrivial Galois automorphism of the field $\mathbb{Q}(\xi)$.

Then the sequence $(a_n)_{n\in\mathbb{N}}$ is eventually periodic, and either all partial quotients in the period belong to $\mathbb{Z}$, or they all belong to $\sqrt{D}\mathbb{Z}$.

Moreover, there is an effective constant C_ξ, which bounds from above the lengths of the period and of the pre-period.

If every partial quotient a_i belongs to $\mathbb{Z}$, we recover the statement of the classical theorem of Lagrange. Theorem 1.1, however, also applies to many other representations of ξ as a continued fraction with partial quotients in a quadratic field. We will present a class of such expansions and we will focus in particular on the β-continued fractions; in this setting, we will apply our theorem to study some open problems on the subject.

We remark that there are, indeed, nontrivial examples in which all partial quotients belong to $\sqrt{D}\mathbb{Z}$, as we will see in Section 6.2.

The proof of Theorem 1.1 is given in Section 3 and relies on diophantine approximation and algebraic number theory. An explicit expression of the bound C_ξ is discussed in Remark 3.4. In the Appendix, we present a more elementary argument, which is nevertheless capable of proving a weaker version of the main theorem.

1.2. A question of Rosen and β-continued fractions. — In 1977, Rosen [18] stated the following research problem: "Is it possible to devise a continued fraction that represents uniquely all real numbers, so that the finite continued fractions represent the elements of an algebraic number field, and conversely, every element of the number field is represented by a finite continued fraction"? The classical regular continued fraction has this property with respect to the field of rational numbers.

For $\lambda = 2\cos\frac{\pi}{q}$ with $q \geq 3$ odd, Rosen gave a definition of λ-continued fractions, whose partial quotients are integral multiples of λ. Rosen showed, as a consequence of his own work [17], that if $q = 5$ (i.e. $\lambda = \varphi = \frac{1}{2}(1+\sqrt{5})$ the golden ratio), the λ-continued fraction satisfies his desired property.

A rather different construction was presented by Bernat [2]. He defined φ-continued fractions whose partial quotients belong to the set of the so-called φ-integers, i.e. numbers whose greedy expansion in base φ uses only non-negative powers of the base. Bernat showed that his φ-continued fractions also represent every element of $\mathbb{Q}(\sqrt{5})$ finitely. His proof is established using a very detailed and tedious analysis of the behaviour of φ-integers under arithmetic operations. This approach crucially depends on the arithmetic properties of φ-integers, descending from the fact that φ is a quadratic Pisot number.

It is natural to ask whether the analogously-defined continued fraction expansion based on the β-integers would provide finite representation of $\mathbb{Q}(\beta)$ for any other choice of a quadratic Pisot number β. When trying to adapt Bernat's proof to other values of β, already in the case of the next smallest quadratic Pisot number $\beta = 1+\sqrt{2}$, the necessary analysis becomes even more technical, preventing one from proving the finiteness of the expansions. Bernat remarked that we do not even know whether for some other value of β other than φ, this construction provides at least an eventually periodic representation of all elements of $\mathbb{Q}(\beta)$.

In this paper, we have taken a different approach, considering more general continued fractions whose partial quotients belong to some discrete subset M of the ring of integers in a real quadratic field K. The β-continued fraction of Bernat is a special case of such M-continued fractions when M is chosen to be the set $\mathbb{Z}_\beta$ of β-integers (see Section 5). With the aim of answering Bernat's problem and classifying all quadratic numbers β according to the qualitative behaviour of the β-continued fraction expansion of elements of $\mathbb{Q}(\beta)$, we introduce the following definitions.

Let $\beta > 1$ be a real algebraic integer.

- (CFF) We say that β has the continued fraction finiteness property (CFF), if every element of $\mathbb{Q}(\beta)$ has a finite β-continued fraction expansion.
- (CFP) We say that β has the continued fraction periodicity property (CFP), if every element of $\mathbb{Q}(\beta)$ has a finite or eventually periodic β-continued fraction expansion.

We show that all quadratic Perron numbers and the square roots of positive integers satisfy (CFP) (Theorems 6.4 and 6.9). We prove (CFF) for four quadratic Perron numbers, including the golden ratio φ (Theorem 6.5). Moreover, assuming a conjecture stated by Mercat [14], we show that these four Perron numbers are the only ones with (CFF).

We also consider the case of non-Perron quadratic β. In this case, if the algebraic conjugate of β is positive, we are able to construct a class of elements in $\mathbb{Q}(\beta)$ having aperiodic β-continued fraction expansion, thus showing that neither (CFF) nor (CFP) hold (Theorem 6.10). To achieve a full characterisation of the numbers for which neither (CFF) nor (CFP) hold, it remains to give a complete answer when β is a non-Perron number with a negative algebraic conjugate. Computer experiments suggest that (CFF) is not satisfied by any such β, with some of them having (CFP) and some not. In Theorem 6.14, we are able to construct counterexamples to (CFF) for a large class of such β, but even assuming Mercat's conjecture the matter is not fully settled.

The full picture of what we are able to prove is summarized in Table 1.1, where $\beta > 1$ is a quadratic integer, and β' is its algebraic conjugate.

Table 1.1. Entries in black are unconditional, entries in blue assume the validity of Mercat's conjecture.

	(CFP)	(CFF)
$\beta' > \beta$	NO (Theorem 6.10)	NO (Theorem 6.10)
$\lvert\beta'\rvert < \beta$	YES (Theorem 6.4)	YES if $\beta = \frac{1+\sqrt{5}}{2}, 1+\sqrt{2}, \frac{1+\sqrt{13}}{2}, \frac{1+\sqrt{17}}{2}$ NO otherwise (Theorem 6.5 and Corollary 6.7)
$\beta' = -\beta$	YES (Theorem 6.9)	NO (Theorem 6.9)
$-\beta - 3 \leq \beta' < -\beta$	Open	Probably NO but still open for 20 values of β (Table 6.2)
$\beta' \leq -\beta - 4$	Open	NO (Theorem 6.14)

These theorems on β-continued fractions are obtained with a wide range of different techniques, ranging from diophantine approximation to algebraic number theory and dynamics.

1.3. Further lines of inquiry. — One could also study the properties (CFF), (CFP) for algebraic integers β of degree bigger than 2. The issue is considerably more intricate, and already in the cubic Pisot case, we can find instances of different β's that seem to have (CFF), (CFP), and aperiodic β-continued fraction expansions. A modification of some of our arguments may be possible in the case of higher degree fields, but its application is likely to be non-trivial.

Another setting in which the same problem might be stated is that of function fields; a function field analogue of β-continued fractions was studied in [11].

We also mention, as a different path of inquiry that could lead to the study of M-continued fractions for other choices of M, the recent work [4]. There, the authors studied periodic continued fractions with fixed lengths of pre-period and period and with partial quotients in the ring of S-integers of a number field, describing such continued fractions as S-integral points on some suitable affine varieties.

2. Preliminaries

We will denote by $\mathbb{N}$ the set of non-negative integers.

2.1. Continued fractions. — Let $(a_i)_{i\in\mathbb{N}}$ be a sequence of real numbers, such that $a_i > 0$ for $i \geq 1$, and define two sequences $(p_n)_{n\geq -2}$, $(q_n)_{n\geq -2}$ by the linear second-order recurrences

$$\begin{aligned} p_n &= a_n p_{n-1} + p_{n-2}, \quad p_{-1} = 1,\ p_{-2} = 0\,, \\ q_n &= a_n q_{n-1} + q_{n-2}, \quad q_{-1} = 0,\ q_{-2} = 1\,. \end{aligned} \tag{1}$$

This recurrence can be written in matrix form as

$$\begin{pmatrix} p_{n+1} & p_n \\ q_{n+1} & q_n \end{pmatrix} = \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix} \begin{pmatrix} a_{n+1} & 1 \\ 1 & 0 \end{pmatrix}, \quad n \geq -1\,.$$

Taking determinants, it can be easily shown that

$$p_{n-1}q_n - p_n q_{n-1} = (-1)^n, \quad n \geq -1\,. \tag{2}$$

By induction, one can also show that

$$[a_0, a_1, \ldots, a_n] := a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}} = \frac{p_n}{q_n}\,.$$

Notice that the assumption of positivity of a_i ensures that the expression on the right-hand side is well defined for every n.