Bull. Soc. Math. France
150 (1), 2022, p. 209-249

A STUDY OF NEFNESS IN HIGHER CODIMENSION

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ABSTRACT. — In this work, following the fundamental work of Boucksom, we construct the nef cone of a compact complex manifold in higher codimension and give explicit examples for which these cones are different. In the third and fourth sections, we give different versions of Kawamata–Viehweg vanishing theorems regarding nefness in higher codimension and numerical dimensions. We also show through examples the optimality of the divisorial Zariski decomposition given in [5].

RÉSUMÉ (*Une étude de l'effectivité numérique en codimension supérieure*). — Dans ce travail, à la suite des travaux fondamentaux de Boucksom, nous construisons le cône nef d'une variété complexe compacte de codimension supérieure et donnons des exemples explicites pour lesquels ces cônes sont différents. Dans les troisième et quatrième sections, nous donnons différentes versions des théorèmes d'annulation de Kawamata-Viehweg en termes de l'effectivité numérique en codimension supérieure et des dimensions numériques. Nous montrons aussi par des exemples l'optimalité de la décomposition divisoriale de Zariski donnée dans [5].

Texte reçu le 28 janvier 2021, modifié le 12 novembre 2021, accepté le 10 janvier 2022.

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Mathematical subject classification (2010). — 32F32, 32Q15, 32U40.

Key words and phrases. — Nefness in higher codimension, numerical dimension, Kawamata–Viehweg vanishing theorem.

This work is supported by the PhD program AMX of École Polytechnique and Ministère de l'Enseignement Supérieur et de la Recherche et de l'Innovation and the European Research Council grant ALKAGE number 670846 managed by J.-P. Demailly.

BULLETIN DE LA SOCIÉTÉ MATHÉMATIQUE DE FRANCE 0037-9484/2022/209/$ 5.00

doi:10.24033/bsmf.2847

1. Introduction

One of the reformulations of the Kodaira embedding theorem is that a compact complex manifold is projective if and only if the Kähler cone, i.e. the convex cone spanned by Kähler forms in $H^2(X,\mathbb{R})$, contains a rational point (i.e. an element in $H^2(X,\mathbb{Q})$).

As a general matter of fact, it is obviously interesting to study positive cones attached to compact complex manifolds and relate them with the geometry of the manifold. In classical algebraic or complex geometry, the emphasis is on two types of positive cones: the nef and psef cones, defined as the closed convex cones spanned by nef classes and psef classes, respectively. The nef cone is, of course, contained in the psef cone.

The work of Boucksom [5] defines and studies the so-called modified nef cone for an arbitrary compact complex manifold. Due to this definition, Boucksom was able to show the existence of a divisorial Zariski decomposition for any psef class (i.e. any cohomology class containing a positive current). The modified cone just sits between the nef and psef cones.

Inspired by Boucksom's definition, in Section 2, we introduce the concept of a nef cone in arbitrary codimension for any compact complex manifold, which is an interpolation between the above positive cones.

Definition A. — Let $\alpha \in H^{1,1}_{BC}(X,\mathbb{R})$ be a psef class. We say that α is nef in codimension k, if for any irreducible analytic subset $Z \subset X$ of codimension at most equal to k, we have the generic minimal multiplicity of α along Z as (defined in [5])

$$\nu(\alpha, Z) = 0.$$

With this terminology, the nef cone is the nef cone in codimension n, where n is the complex dimension of the manifold, while the psef cone is the nef cone in codimension 0, and the modified nef cone is the nef cone in codimension 1. We notice that the algebraic analogue in the projective case is introduced in [32]. In Section 4, we show that these cones are, in general, different and construct explicit examples where they are different.

Inspired by the work of [9] and using Guan–Zhou's solution of Demailly's strong openness conjecture, we get the following Kawamata–Viehweg vanishing theorem in Section 3. The proof follows Cao's proof closely:

Theorem A. — *Let (L,h) be a pseudo-effective line bundle on a compact Kähler n-dimensional manifold X with singular positive metric h. Then the morphism induced by the inclusion $K_X \otimes L \otimes \mathcal{I}(h) \to K_X \otimes L$*

$$H^q(X, K_X \otimes L \otimes \mathcal{I}(h)) \to H^q(X, K_X \otimes L)$$

vanishes for every $q \geq n - \mathrm{nd}(L) + 1$.

As an application, in Section 4, we obtain the following generalisation from the nef case to the psef case of a similar result stated in [16].

Theorem B. — *Let (X, ω) be a compact Kähler manifold of dimension n and L a line bundle on X that is nef in codimension 1. Assume that $\langle L^2 \rangle \neq 0$ where $\langle \cdot \rangle$ is the positive product defined in* [4]*. Assume that there exists an effective integral divisor D such that $c_1(L) = c_1(D)$. Then*

$$H^q(X, K_X + L) = 0,$$

for $q \geq n - 1$.

The proof of the above theorem is an induction on the dimension, using Theorem A. A difference compared with the nef case treated in [16] is that instead of an intersection number we need to use a positive product (or movable intersection number), which is a non-linear operation. Nevertheless, under a condition of nefness in higher codimension, we get the following estimate.

Lemma A. — *Let α be a nef class in codimension p on a compact Kähler manifold (X, ω), then for any $k \leq p$ and Θ any positive closed $(n-k, n-k)$-form, we have*

$$(\alpha^k, \Theta) \geq \langle \alpha^k, \Theta \rangle.$$

With this inequality, the intersection number calculation in [16] is still valid, and thus the cohomology calculations can be recycled.

Observe that a current with minimal singularities need not have analytic singularities for every big class α that is nef in codimension 1 but not nef in codimension 2; such an example was given by [32], and also observed by Matsumura [30].

As a consequence of Matsumura's observation, the assumption of our Kawamata–Viehweg vanishing theorem that the line bundle is numerically equivalent to an effective integral divisor is actually required. In the nef case considered in [16], the authors deduce from their assumption that the line bundle L is nef with $(L^2) \neq 0$ that L is numerically equivalent to an effective integral divisor D, and that there exists a positive singular metric h on L, such that $\mathcal{I}(h) = \mathcal{O}(-D)$.

However, for a big line bundle L that is nef in codimension 1 but not nef in codimension 2 over an arbitrary compact Kähler manifold (X, ω), we have that $\langle L^2 \rangle \neq 0$ and $\frac{i}{2\pi}\Theta(L, h_{\min})$ need not be a current associated with an effective integral divisor.

Another by-product is the (probably already known) example of a projective manifold X with $-K_X$ psef, for which the Albanese morphism is not surjective. It was proven in [8], [34] (and [38] for the projective case) that the Albanese morphism of a compact Kähler manifold with $-K_X$ nef is always surjective. Thus, replacing nefness by pseudo-effectivity in the study of Albanese morphism seems to be a non-trivial problem.

2. Nefness in higher codimension

We first recall some technical preliminaries introduced in [5]. Throughout this paper, X is assumed to be a compact complex manifold equipped with some reference Hermitian metric ω (i.e. a smooth positive definite $(1,1)$-form); we usually take ω to be Kähler, if X possesses such metrics. The Bott–Chern cohomology group $H^{1,1}_{BC}(X,\mathbb{R})$ is the space of d-closed smooth (1,1)-forms modulo $i\partial\overline{\partial}$-exact ones. By the quasi-isomorphism induced by the inclusion of smooth forms into currents, $H^{1,1}_{BC}(X,\mathbb{R})$ can also be seen as the space of d-closed $(1,1)$-currents modulo $i\partial\overline{\partial}$-exact ones. A cohomology class $\alpha \in H^{1,1}_{BC}(X,\mathbb{R})$ is said to be pseudo-effective iff it contains a positive current; α is nef iff, for each $\varepsilon > 0$, α contains a smooth form α_ε, such that $\alpha_\varepsilon \geq -\varepsilon\omega$; α is big iff it contains a Kähler current, i.e. a closed $(1,1)$-current T, such that $T \geq \varepsilon\omega$ for $\varepsilon > 0$ small enough.

Definition 2.1 ([19]). — Let φ_1, φ_2 be two quasi-psh functions on X (i.e. $i\partial\overline{\partial}\varphi_i \geq -C\omega$ in the sense of currents for some $C \geq 0$). The function φ_1 is said to be less singular than φ_2 (one then writes $\varphi_1 \preceq \varphi_2$) if $\varphi_2 \leq \varphi_1 + C_1$ for some constant C_1. Let α be a fixed psef class in $H^{1,1}_{BC}(X,\mathbb{R})$. Given $T_1, T_2, \theta \in \alpha$ with θ smooth, and $T_i = \theta + i\partial\overline{\partial}\varphi_i$ with φ_i quasi-psh $(i = 1,2)$, we write $T_1 \preceq T_2$ iff $\varphi_1 \preceq \varphi_2$ (notice that for any choice of θ, the potentials φ_i are defined up to smooth bounded functions, since X is compact). If γ is a smooth real $(1,1)$-form on X, the collection of all potentials φ, such that $\theta + i\partial\overline{\partial}\varphi \geq \gamma$ admits a minimal element $T_{min,\gamma}$ for the pre-order relation $\preceq$, constructed as the semi-continuous upper envelope of the sub-family of potentials $\varphi \leq 0$ in the collection.

Definition 2.2 (Minimal multiplicities). — The minimal multiplicity at $x \in X$ of the pseudo-effective class $\alpha \in H^{1,1}_{BC}(X,\mathbb{R})$ is defined as

$$\nu(\alpha, x) := \sup_{\varepsilon > 0} \nu(T_{\min,\varepsilon}, x),$$

where $T_{\min,\varepsilon}$ is the minimal element $T_{\min,-\varepsilon\omega}$ in the above definition, and $\nu(T_{\min,\varepsilon}, x)$ is the Lelong number of $T_{\min,\varepsilon}$ at x. When Z is an irreducible analytic subset, we define the generic minimal multiplicity of α along Z as

$$\nu(\alpha, Z) := \inf\{\nu(\alpha, x), x \in Z\}.$$

When Z is positive dimensional, there exists for each $\ell \in \mathbb{N}^*$ a countable union of proper analytic subsets of Z denoted by $Z_\ell = \bigcup_p Z_{\ell,p}$, such that $\nu(T_{\min,\frac{1}{\ell}}, Z) := \inf_{x\in Z} \nu(T_{\min,\frac{1}{\ell}}, x) = \nu(T_{\min,\frac{1}{\ell}}, x)$, for $x \in Z \smallsetminus Z_\ell$. By construction, when $\varepsilon_1 < \varepsilon_2$, $T_{\min,\varepsilon_1} \succeq T_{\min,\varepsilon_2}$. Hence, for a very general point $x \in Z \smallsetminus \bigcup_{\ell\in\mathbb{N}^*} Z_\ell$,

$$\nu(\alpha, Z) \leq \nu(\alpha, x) = \sup_{\ell} \nu(T_{\min,\frac{1}{\ell}}, Z).$$

On the other hand, for any $y \in Z$,

$$\sup_{\ell} \nu(T_{\min,\frac{1}{\ell}}, Z) \leq \sup_{\ell} \nu(T_{\min,\frac{1}{\ell}}, y) = \nu(\alpha, y).$$

In conclusion, $\nu(\alpha, Z) = \nu(\alpha, x)$, for a very general point $x \in Z \smallsetminus \bigcup_{\ell \in \mathbb{N}^*} Z_\ell$, and $\nu(\alpha, Z) = \sup_\varepsilon \nu(T_{\min,\varepsilon}, Z)$.

Now we can define the concept of nefness in higher codimension implicitly used in [5]. It is the generalisation of the concept of "modified nefness" to the higher codimensional case.

Definition 2.3. — Let $\alpha \in H^{1,1}_{BC}(X, \mathbb{R})$ be a psef class. We say that α is nef in codimension k, if for every irreducible analytic subset $Z \subset X$ of codimension at most equal to k, we have

$$\nu(\alpha, Z) = 0.$$

We denote by $\mathcal{N}_k$ the cone generated by nef classes in codimension k. By Proposition 3.2 in [5], a psef class α is nef, iff for any $x \in X$, $\nu(\alpha, x) = 0$. By our definition, psef is equivalent to nef in codimension 0, and nef is equivalent to nef in codimension $n := \dim_{\mathbb{C}} X$. In this way, we get a bunch of positive cones on X, satisfying the inclusion relations

$$\mathcal{N} = \mathcal{N}_n \subset \cdots \subset \mathcal{N}_1 \subset \mathcal{N}_0 = \mathcal{E},$$

where $\mathcal{N}$ and $\mathcal{E}$ are cones of nef and psef classes, respectively. By a proof similar to those of Propositions 3.5 and 3.6 in [5], we get:

Proposition 2.4. —
(1) *For every $x \in X$ and every irreducible analytic subset Z, the map $\mathcal{E} \to \mathbb{R}^+$ defined on the cone $\mathcal{E}$ of psef classes by $\alpha \mapsto \nu(\alpha, Z)$ is convex and homogeneous. It is continuous on the interior $\mathcal{E}^\circ$ and lower semi-continuous on the whole of $\mathcal{E}$.*
(2) *If $T_{\min} \in \alpha$ is a positive current with minimal singularities, we have $\nu(\alpha, Z) \leq \nu(T_{\min}, Z)$.*
(3) *If α is moreover big, we have $\nu(\alpha, Z) = \nu(T_{\min}, Z)$.*

The following lemma is a direct application of the proposition.

Lemma 2.5. — *Let $Y \subset X$ be a smooth sub-manifold of X and $\pi : \tilde{X} \to X$ be the blow-up of X along Y. We denote by E the exceptional divisor. If $\alpha \in H^{1,1}_{BC}(X, \mathbb{R})$ is a big class, we have*

$$\nu(\alpha, Y) = \nu(\pi^* \alpha, E).$$

For Z any irreducible analytic set not included in Y, we denote by $\tilde{Z}$ the strict transform of Z. Then

$$\nu(\alpha, Z) = \nu(\pi^* \alpha, \tilde{Z}).$$