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CLASSIFICATION OF THOMPSON RELATED GROUPS ARISING FROM JONES' TECHNOLOGY II

BY ARNAUD BROTHIER

ABSTRACT. — In this second article, we continue to study classes of groups constructed from a functorial method due to Vaughan Jones. A key observation of the author shows that these groups admit remarkable diagrammatic descriptions that can be used to deduce their properties. Given any group and two of its endomorphisms, we construct a semi-direct product. In our first article dedicated to this construction, we classify up to isomorphism all these semi-direct products when one of the endomorphisms is trivial and describe their automorphism group.

In this article, we focus on the case where both endomorphisms are automorphisms. The situation is rather different, and we obtain semi-direct products where the largest Richard Thompson's group V is acting on some discrete analogues of loop groups. Note that these semi-direct products appear naturally in recent constructions of quantum field theories. Moreover, they were previously studied by Tanushevski and can be constructed via the framework of cloning systems of Witzel–Zaremsky. In particular, they provide examples of groups with various finiteness properties and possible counter-examples of a conjecture of Lehnert on co-context-free groups.

We provide a partial classification of these semi-direct products and describe their automorphism group explicitly. Moreover, we prove that groups studied in the first and second articles are never isomorphic to each other nor do they admit nice embeddings

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between them. We end the article with an appendix comparing Jones' technology with Witzel–Zaremsky's cloning systems and with Tanushevski's construction. As in the first article, it was possible to achieve all the results presented via a surprising rigidity phenomenon on isomorphisms between these groups.

RÉSUMÉ (*Classification de groupes reliés à celui de Thompson et construits à l'aide d'une technologie de Jones II*). — Dans ce second article nous continuons d'étudier des classes de groupes construites à l'aide d'une méthode fonctorielle de Vaughan Jones. Une observation clef de l'auteur montre que ces groupes admettent des descriptions diagrammatiques remarquables qui peuvent être utilisées pour en déduire leurs propriétés. Étant donné un groupe quelconque et deux de ses endomorphismes nous construisons un produit semi-direct. Dans notre premier article dédié à cette construction nous avons classifié à isomorphisme près tous ces produits semi-directs lorsque l'un des endomorphismes est trivial et avons décrit leur groupe d'automorphisme.

Dans cet article, nous nous concentrons sur le cas où les deux endomorphismes sont des automorphismes. Cette situation est très différente et l'on obtient des produits semi-directs où le plus grand groupe de Richard Thompson agit sur un analogue discret d'un groupe de lacets. Notons que ces produits semi-directs apparaissent naturellement dans de récentes constructions de théories quantiques des champs. De plus, ils ont été précédemment étudiés par Tanushevski et peuvent être construits à l'aide des systèmes de clonage de Witzel-Zaremsky. En particulier, ils donnent des exemples de groupes satisfaisant différentes propriétés de finitude et une classe de contre-exemples potentiels à une conjecture de Lehnert portant sur les groupes ayant un co-langage non-contextuel.

Nous donnons une classification partielle de ces produits semi-directs et décrivons explicitement leur groupe d'automorphisme. De plus, nous montrons qu'il n'y a pas d'isomorphismes ou même de bonnes injections entre un groupe étudié dans le premier article et un groupe étudié dans le deuxième. Nous terminons par un appendice qui compare la technologie de Jones avec les systèmes de clonage de Witzel-Zaremsky et la construction de Tanushevski. Comme dans le premier article, tous les résultats présenté ont été obtenu grâce à un phénomène de rigidité surprenant portant sur les isomorphismes entre ces groupes.

Introduction

Jones' technology. — Jones' subfactor theory has been intimately linked with chiral conformal field theory (in short CFT) for more than two decades. Longo and Rehren proved that any CFT produces a subfactor [37]. Conversely, certain subfactors produce CFT, but there is no general construction [4]. This led Vaughan Jones to ask the fundamental question: "Does every subfactor have something to do with a conformal field theory?" Over several decades, Jones tried to answer this question by looking for a general construction that would associate a CFT to *any* subfactor. In his last attempt, Jones constructed a discretisation of a CFT from a subfactor, where Thompson group T replaced the usual conformal group invariance [31]. The idea was to take a limit and obtain an honest classical CFT. However, discontinuity issues appeared, and the CFT goal stayed out of touch [32].

Jones realised that the technology he developed to construct a discrete CFT was very useful and practical for constructing group actions [12]. Given a nice category $\mathcal{C}$ (a category with a left or right calculus of fractions, e.g. a commutative cancellative monoid), there is a well-known process for constructing a so-called fraction group $G_{\mathcal{C}}$ (in fact, a fraction groupoid) from this category [26, Section 1]. Jones discovered that given *any* functor $\Phi : \mathcal{C} \to \mathcal{D}$ ending in *any* category $\mathcal{D}$, one can construct very explicitly an action (called a *Jones action*) $\pi_\Phi : G_{\mathcal{C}} \curvearrowright K_\Phi$ of the fraction group $G_{\mathcal{C}}$ of the source category $\mathcal{C}$ on a set K_Φ (sometimes K_Φ will rather be an object in a category) constructed from the functor Φ. Moreover, properties of the target category $\mathcal{D}$ are reflected in the properties of the Jones action π_Φ. For instance, a functor $\Phi : \mathcal{C} \to \mathrm{Hilb}$ ending in the category of Hilbert spaces with isometries for morphisms provides a unitary representation π_Φ of the fraction group $G_{\mathcal{C}}$. We say that π_Φ is a *Jones representation.* A functor $\Phi : \mathcal{C} \to \mathrm{Gr}$ ending in the category of groups provides an action by group automorphisms $\pi_\Phi : G_{\mathcal{C}} \curvearrowright K_\Phi$ on a group K_Φ.

A key example that we will be considering in this article is given by the category $\mathcal{C} = \mathcal{F}$ of (rooted, finite, ordered, binary) forests whose fraction group is Richard Thompson's group F; the group of piecewise linear homeomorphisms of the unit interval having finitely many breakpoints at dyadic rationals and slopes of the power of 2 [18]. Moreover, considering larger categories of forests equipped with permutations of their leaves, one can build the larger Thompson groups T and V as fraction groups that are the analogues of F but acting by homeomorphisms on the unit torus and on the Cantor space, respectively. Recall that we have the chain of inclusions $F \subset T \subset V$.

Applications of Jones' technology. — Jones' technology for producing group actions is recent (less than 10 year's old) but has already provided a number of interesting applications and new connections between fields of mathematics. Functors $\Phi : \mathcal{F} \to \mathcal{D}$ from the category of forests going to a Jones planar algebra $\mathcal{D}$ produce discrete CFT [31, 32]. These are physical field theories that are sometimes called Thompson field theory and are now studied in their own right [33, 40, 44]. This led to fruitful developments in operator algebraic quantum field theory [17, 16]. Those functors produce, among others, new and interesting unitary representations of the Thompson groups using the inner product structure given by the quantum trace [34, 2]. For instance, Jones provided a one-parameter family of irreducible unitary representations of F that are pairwise non-isomorphic and where the parameter of the family is the celebrated index set of subfactors $\{4\cos(\pi/n) : \; n \geq 3\} \cup [4, \infty)$ [34].

Functors $\Phi : \mathcal{F} \to \mathcal{D}$ ending in the category of Conway tangles provide a way to construct knots and links from Thompson group elements [31]. This is a fascinating and deep connection between Thompson group F, knot theory and