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ON THE STRUCTURAL THEORY OF II_1 FACTORS OF NEGATIVELY CURVED GROUPS

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ABSTRACT. – Ozawa showed in [21] that for any i.c.c. hyperbolic group, the associated group factor $L\Gamma$ is solid. Developing a new approach that combines some methods of Peterson [29], Ozawa and Popa [27, 28], and Ozawa [25], we strengthen this result by showing that $L\Gamma$ is strongly solid. Using our methods in cooperation with a cocycle superrigidity result of Ioana [12], we show that profinite actions of lattices in $\mathrm{Sp}(n,1)$, $n \geq 2$, are virtually W^*-superrigid.

RÉSUMÉ. – Ozawa a montré dans [21] que, pour un groupe c.c.i. hyperbolique, le facteur de type II_1 associé est solide. En développant une nouvelle approche, qui combine les méthodes de Peterson [29], d'Ozawa et Popa [27, 28], et d'Ozawa [25], nous renforçons ce résultat en montrant que ce facteur est fortement solide. En combinant nos méthodes avec un résultat d'Ioana de superrigidité des cocycles [12], nous prouvons que les actions des réseaux de $\mathrm{Sp}(n,1)$, $n \geq 2$, sont virtuellement W^*-superrigides.

Introduction

In a conceptual leap Ozawa established a broad property for group factors of Gromov hyperbolic groups—what he termed solidity—which essentially allowed him to reflect the "small cancellation" property such a group enjoys in terms of its associated von Neumann algebra.

OZAWA'S SOLIDITY THEOREM ([21]). – *If Γ is an i.c.c. Gromov hyperbolic group, then $L\Gamma$ is solid, i.e., $A' \cap L\Gamma$ is amenable for every diffuse von Neumann subalgebra $A \subset L\Gamma$.*

Notable for its generality, Ozawa's argument relies on a surprising interplay between C^*-algebraic and von Neumann algebraic techniques [4].

Using his deformation/rigidity theory [32], Popa was able to offer an alternate, elementary proof of solidity for free group factors: more generally, for factors admitting a "free malleable deformation" [33]. Popa's approach exemplifies the use of spectral gap rigidity arguments that opened up many new directions in deformation/rigidity theory, cf. [32, 33, 34]. Of

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particular importance, these techniques brought the necessary perspective for a remarkable new approach to the Cartan problem for free group factors in the work of Ozawa and Popa [27, 28]—an approach which this work directly builds upon.

A new von Neumann-algebraic approach to solidity was developed by Peterson in his important paper on L^2-rigidity [29]. Essentially, Peterson was able to exploit the "negative curvature" of the free group on two generators $\mathbb{F}_2$, in terms of a proper 1-cocycle into the left-regular representation, to rule out the existence of large relative commutants of diffuse subalgebras of $L\mathbb{F}_2$.

PETERSON'S SOLIDITY THEOREM ([29]). – *If Γ is an i.c.c. countable discrete group which admits a proper 1-cocycle $b : \Gamma \to \mathcal{H}_\pi$ for some unitary representation π which is weakly-ℓ^2 (i.e., weakly contained in the left-regular representation), then $L\Gamma$ is solid.*

It was later realized by the second author [39] that many of the explicit unbounded derivations (i.e., the ones constructed from 1-cocycles) that Peterson works with have natural dilations which are malleable deformations of their corresponding (group) von Neumann algebras.

However, the non-vanishing of 1-cohomology of Γ with coefficients in the left-regular representation does not reflect the full spectrum of negative curvature phenomena in geometric group theory as evidenced by the existence of non-elementary hyperbolic groups with Kazhdan's property (T), cf. [3]. In their fundamental works on the rigidity of group actions [17, 18], Monod and Shalom proposed a more inclusive cohomological definition of negative curvature in group theory which is given in terms of non-vanishing of the second-degree bounded cohomology for Γ with coefficients in the left-regular representation. Relying on Monod's work in bounded cohomology [16], we will make use of a related condition, which is the existence of a proper *quasi*-1-cocycle on Γ into the left-regular representation (more generally, into a representation weakly contained in the left-regular representation), cf. [16, 40]. By a result of Mineyev, Monod, and Shalom [15], this condition is satisfied for any hyperbolic group—the case of vanishing first ℓ^2-Betti number is due to Mineyev [14].

Statement of results

We now state the main results of the paper, in order to place them within the context of previous results in the structural theory of group von Neumann algebras. We begin with the motivating result of the paper, which unifies the solidity theorems of Ozawa and Peterson.

THEOREM A. – *Let Γ be an i.c.c. countable discrete group which is exact and admits a proper quasi-1-cocycle $q : \Gamma \to \mathcal{H}_\pi$ for some weakly-ℓ^2 unitary representation π (more generally, Γ is exact and belongs to the class $\mathcal{QH}_{\rm reg}$ of Definition 1.6). Then $L\Gamma$ is solid.*

In particular, all Gromov hyperbolic groups are exact, cf. [36], and admit a proper quasi-1-cocycle for the left-regular representation [15]. For the class of exact groups, belonging to the class $\mathcal{QH}_{\rm reg}$ is equivalent to bi-exactness (see Section 1), so the above result is equivalent to Ozawa's Solidity Theorem.

Following Ozawa's and Peterson's work on solidity, there was some hope that similar techniques could be used to approach to the Cartan subalgebra problem for group factors of hyperbolic groups, generalizing Voiculescu's celebrated theorem on the absence of

Cartan subalgebras for free group factors [43]. However, the Cartan problem for general hyperbolic groups would remain intractable until the breakthrough approach of Ozawa and Popa through Popa's deformation/rigidity theory resolved it in the positive for the group factor of any discrete group of isometries of the hyperbolic plane [28]. In fact, they were able to show that any such II_1 factor M is *strongly solid*, i.e., for every diffuse, amenable von Neumann subalgebra $A \subset M$, $\mathcal{N}_M(A)'' \subset M$ is an amenable von Neumann algebra, where $\mathcal{N}_M(A) = \{u \in \mathcal{U}(M) : uAu^* = A\}$.

Using the techniques developed by Ozawa and Popa [27, 28] and a recent result of Ozawa [25], we obtain the following strengthening of Theorem A.

Theorem B. – *Let Γ be an i.c.c. countable discrete group which is weakly amenable (therefore, exact). If Γ admits a proper quasi-1-cocycle into a weakly-ℓ^2 representation, then $L\Gamma$ is strongly solid.*

Appealing to Ozawa's proof of the weak amenability of hyperbolic groups [23], Theorem B allows us to fully resolve in the positive the strong solidity problem—hence the Cartan problem—for i.c.c. hyperbolic groups and for lattices in connected rank one simple Lie groups. In particular, if Γ is an i.c.c. lattice in $\mathrm{Sp}(n,1)$ or the exceptional group $\mathrm{F}_{4(-20)}$, then $L\Gamma$ is strongly solid. The strong solidity problem for the other rank one simple Lie groups—those locally isomorphic to $\mathrm{SO}(n,1)$ or $\mathrm{SU}(n,1)$)—was resolved for $\mathrm{SO}(2,1)$, $\mathrm{SO}(3,1)$, and $\mathrm{SU}(1,1)$ by the work of Ozawa and Popa [28] and, in the general case, by the work of the second author [39]. The results follow directly from Theorem B in the co-compact (i.e., uniform) case: in the non-uniform case, we must appeal to a result of Shalom (Theorem 3.7 in [38]) on the integrability of lattices in connected simple rank one Lie groups to produce a proper quasi-1-cocycle.

Building on Ioana's work on cocycle superrigidity [12], we are also able to obtain new examples of virtually W^*-superrigid actions.

Corollary B.1. – *Let Γ be an i.c.c. countable discrete group which is weakly amenable and which admits a proper quasi-1-cocycle into a weakly-ℓ^2 representation. If $\Gamma \curvearrowright (X,\mu)$ is a profinite, free, ergodic measure-preserving action of Γ on a standard probability space (X,μ), then $L^\infty(X,\mu) \rtimes \Gamma$ has a unique Cartan subalgebra up to unitary conjugacy. If in addition Γ has Kazhdan's property (T) (e.g., Γ is a lattice in $\mathrm{Sp}(n,1)$, $n \geq 2$), then any such action $\Gamma \curvearrowright (X,\mu)$ is virtually W^*-superrigid.*

A natural question to ask is whether our techniques can be extended to demonstrate strong solidity of the group factor of any i.c.c. countable discrete group which is relatively hyperbolic [19] to a family of amenable subgroups.

The techniques used to prove Theorem B also allow us to deduce, by way of results of Cowling and Zimmer [8] and Ozawa [23], the following improvement of results of Adams (Corollary 6.2 in [1]) and of Monod and Shalom (Corollary 1.19 in [18]) on the structure of groups which are orbit equivalent to hyperbolic groups.

Corollary B.2. – *Let Γ be an i.c.c. countable discrete group which is weakly amenable and which admits a proper quasi-1-cocycle into a weakly-ℓ^2 representation. Let $\Gamma \curvearrowright (X,\mu)$ be a free, ergodic, measure-preserving action of Γ on a probability space and $\Lambda \curvearrowright (Y,\nu)$ be* *an arbitrary free, ergodic, measure-preserving action of some countable discrete group Λ on a probability space. If $\Gamma \curvearrowright (X,\mu)$ is orbit equivalent to $\Lambda \curvearrowright (Y,\nu)$, then Λ is not isomorphic to a non-trivial direct product of infinite groups and the normalizer of any infinite, amenable subgroup $\Sigma < \Lambda$ is amenable.*

Beyond solidity results, we highlight that the techniques developed in this paper also enable us to reprove strong decomposition results for products of groups in the spirit of Popa's deformation/rigidity theory. Specifically, we are able to recover the following prime decomposition theorem of Ozawa and Popa.

THEOREM C (Ozawa and Popa [26]). – *Let $\Gamma = \Gamma_1 \times \cdots \times \Gamma_n$ be a non-trivial product of exact, i.c.c. countable discrete groups such that $\Gamma_i \in \mathcal{QH}_{\rm reg}$, $1 \le i \le n$. If $N = N_1 \overline{\otimes} \cdots \overline{\otimes} N_m$ is a product of II_1 factors N_j, $1 \le j \le m$, for some $m \ge n$, and $L\Gamma \cong N$, then $m = n$ and there exist $t_1, \dots, t_n > 0$ with $t_1 \cdots t_n = 1$ so that, up to a permutation, $(L\Gamma_i)^{t_i} \cong N_i$, $1 \le i \le n$.*

An advantage to our approach is that our proof naturally generalizes to unique measure-equivalence decomposition of products of bi-exact groups, first proven by Sako (Theorem 4 in [37]). This type of result was first achieved for products of groups of the class $\mathcal{C}_{\rm reg}$ by Monod and Shalom (Theorem 1.16 in [18]).

COROLLARY C (Sako [37]). – *Let $\Gamma = \Gamma_1 \times \cdots \times \Gamma_n$ be a non-trivial product of exact, i.c.c. countable discrete groups such that $\Gamma_i \in \mathcal{QH}_{\rm reg}$, $1 \le i \le n$, and let $\Lambda = \Lambda_1 \times \cdots \times \Lambda_m$ be a product of arbitrary countably infinite discrete groups. Assume that $\Gamma \sim_{\rm ME} \Lambda$, i.e., there exist $\Gamma \curvearrowright (X,\mu)$ and $\Lambda \curvearrowright (Y,\nu)$ free, ergodic, probability measure-preserving actions which are weakly orbit equivalent (Definition 2.2 in [9]). If $m \ge n$ then $m = n$ and, up to permuting indices, we have that $\Gamma_i \sim_{\rm ME} \Lambda_i$, $1 \le i \le n$.*

On the method of proof

This paper began as an attempt to chart a "middle path" between the solidity theorems of Ozawa, Popa, and Peterson by recasting Ozawa's approach to solidity effectively as a deformation/rigidity argument. We did so by finding a "cohomological" characterization of Ozawa's notion of bi-exactness [22]. Interestingly, our reformulation of bi-exactness has many affinities with (strict) cohomological definitions of negative curvature proposed by Monod and Shalom [18] and Thom [40].

Working from the cohomological perspective, we were able to construct "deformations" of $L\Gamma$. Though these "deformations" no longer mapped $L\Gamma$ into itself, we were still able to control their convergence on a weakly dense C^*-subalgebra of $L\Gamma$ namely, the reduced group C^*-algebra $\mathrm{C}^*_\lambda(\Gamma)$, then borrow Ozawa's insight of using local reflexivity to pass from $\mathrm{C}^*_\lambda(\Gamma)$ to the entire von Neumann algebra.

After this initial undertaking had been completed, we turned our attention to applying these techniques to the foundational methods which Ozawa and Popa developed in proving strong solidity of free group factors. Our approach through deformation/rigidity-type arguments allowed us to exploit the "compactness" of deformations coming from quasi-cocycles to achieve a finer degree of control than is afforded by the use of bi-exactness. This extra control was crucial in our adaptation of Ozawa and Popa's fundamental techniques in the proof of Theorem B. This should be considered the main technical advance of the paper.