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OPENNESS OF UNIFORM K-STABILITY IN FAMILIES OF $\mathbb{Q}$-FANO VARIETIES

BY Harold BLUM AND Yuchen LIU

ABSTRACT. – We show that uniform K-stability is a Zariski open condition in $\mathbb{Q}$-Gorenstein families of $\mathbb{Q}$-Fano varieties. To prove this result, we consider the behavior of the stability threshold in families. The stability threshold (also known as the delta-invariant) is a recently introduced invariant that is known to detect the K-semistability and uniform K-stability of a $\mathbb{Q}$-Fano variety. We show that the stability threshold is lower semicontinuous in families and provides an interpretation of the invariant in terms of the K-stability of log pairs.

RÉSUMÉ. – Nous démontrons que la K-stabilité uniforme est une condition ouverte pour la topologie de Zariski dans une famille $\mathbb{Q}$-Gorenstein de variétés algébriques $\mathbb{Q}$-Fano. Pour prouver ce résultat, nous considérons le seuil de stabilité en familles. Le seuil de stabilité (également appelé l'invariant delta) est un invariant qui détecte la K-semistabilité et la K-stabilité uniforme d'une variété algébrique $\mathbb{Q}$-Fano. Nous démontrons que le seuil de stabilité est semi-continu inférieurement en familles et fournissons une interprétation de cet invariant en termes de K-stabilité des paires log.

Throughout, we work over a characteristic zero algebraically closed field k.

1. Introduction

In this article, we consider the behavior of K-stability in families of $\mathbb{Q}$-Fano varieties. Recall that K-stability is an algebraic notion introduced by Tian [53] and later reformulated by Donaldson [21] to detect the existence of certain canonical metrics on complex projective varieties. In the special case of complex $\mathbb{Q}$-Fano varieties, the Yau-Tian-Donaldson conjecture states that a complex $\mathbb{Q}$-Fano variety is K-polystable iff it admits a Kähler-Einstein metric. (By a $\mathbb{Q}$-Fano variety, we mean a projective variety that has at worst klt singularities and anti-ample canonical divisor.) For smooth complex Fano varieties, this conjecture was recently settled in the work of Chen-Donaldson-Sun and Tian [16, 54] (see also [19, 17, 2]).

Another motivation for understanding the K-stability of $\mathbb{Q}$-Fano varieties is to construct compact moduli spaces for such varieties. It is expected that there is a proper good moduli space parametrizing K-polystable $\mathbb{Q}$-Fano varieties of fixed dimension and volume. For

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smoothable $\mathbb{Q}$-Fano varieties, such a moduli space is known to exist [40] (see also [48, 43]). A key step in constructing the moduli space of K-polystable Fano varieties is verifying the Zariski openness of K-semistability. Towards this goal, we prove

THEOREM A. – *If $\pi : X \to T$ is a projective flat family of varieties such that T is normal, π has normal connected fibers, and $-K_{X/T}$ is $\mathbb{Q}$-Cartier and π-ample, then*

(1) *$\{t \in T \mid X_{\bar{t}}$ is uniformly K-stable$\}$ is a Zariski open subset of T, and*

(2) *$\{t \in T \mid X_{\bar{t}}$ is K-semistable$\}$ is a countable intersection of Zariski open subsets of T.*

The notion of uniform K-stability is a strengthening of K-stability introduced in [11, 20]. In [2], it was shown that a smooth Fano variety X with discrete automorphism group is uniformly K-stable iff there exists a Kähler-Einstein metric on X. The latter equivalence was later extended to $\mathbb{Q}$-Fano varieties with discrete automorphism group in [55]. K-semistability is strictly weaker than K-(poly)stability and corresponds to being almost Kähler-Einstein [37, 2].

In [7], the first author and Xu show that the moduli functor of uniformly K-stable $\mathbb{Q}$-Fano varieties of fixed volume and dimension is represented by a separated Deligne-Mumford stack, which has a coarse moduli space that is a separated algebraic space. The proof of the result combines Theorem A.1 with a boundedness statement in [28] (that uses ideas from [3]) and a separatedness statement in [7].

For smooth families of Fano varieties, Theorem A is not new. Indeed, for a smooth family of complex Fano varieties with discrete automorphism group, the K-stable locus is Zariski open by [44, 23]. In [40], it was shown that the K-semistable locus is Zariski open in families of smoothable $\mathbb{Q}$-Fano varieties. These results all rely on deep analytic tools developed in [16, 54].

Unlike the previous results, our proof of Theorem A is purely algebraic. (A different algebraic proof of Theorem A.2 was also given in [6] using a characterization of K-semistability in terms of the normalized volume of the affine cone over a $\mathbb{Q}$-Fano variety [36, 38, 41].) Furthermore, the result holds for all $\mathbb{Q}$-Fano varieties, including those that are not smooth(able), and also log Fano pairs. The argument relies on new tools for characterizing the uniform K-stability and K-semistability of Fano varieties [11, 36, 24, 25, 5].

More precisely, our approach to proving Theorem A is through understanding the behavior of the *stability threshold* (also known as *δ-invariant* or *basis log canonical threshold*) in families. We recall the definition of this new invariant.

Let X be a projective klt variety and L an ample Cartier divisor on X. Set

$$|L|_{\mathbb{Q}} := \{D \in \mathrm{Div}(X)_{\mathbb{Q}} \mid D \geq 0 \text{ and } mD \sim mL \text{ for some } m \in \mathbb{Z}_{>0}\}.$$

Following [25], we say that $D \in |L|_{\mathbb{Q}}$ is an m-basis type divisor of L if there exists a basis $\{s_1, \ldots, s_{N_m}\}$ of $H^0(X, \mathcal{O}_X(mL))$ such that

$$D = \frac{1}{mN_m}\big(\{s_1 = 0\} + \cdots + \{s_{N_m} = 0\}\big).$$

For $m \in M(L) := \{m \mid h^0(X, \mathcal{O}_X(mL)) \neq 0\}$, set

$$\delta_m(X; L) := \inf_{D\ m\text{-basis type}} \mathrm{lct}(X; D),$$

where $\operatorname{lct}(X; D)$ denotes the log canonical threshold of D. The stability threshold of L is

$$\delta(X; L) := \limsup_{M(L) \ni m \to \infty} \delta_m(X; L).$$

In fact, the above limsup is a limit by [5]. If X is a $\mathbb{Q}$-Fano variety, we set

$$\delta(X) := r\delta(X; -rK_X),$$

where $r \in \mathbb{Z}_{>0}$ is such that $-rK_X$ is Cartier. (The definition is independent of the choice of r.)

The stability threshold is closely related to the *global log canonical threshold of* L, which is an algebraic version of Tian's α-invariant. Recall that the global log canonical threshold of L is

$$\alpha(X; L) := \inf_{D \in |L|_{\mathbb{Q}}} \operatorname{lct}(X; D).$$

The two thresholds satisfy

$$\frac{n+1}{n}\alpha(X; L) \leq \delta(X; L) \leq (n+1)\alpha(X; L),$$

where $n = \dim(X)$.

The stability threshold was introduced in the $\mathbb{Q}$-Fano case by K. Fujita and Y. Odaka to characterize the K-stability of $\mathbb{Q}$-Fano varieties [25]. More generally, the invariant coincides with an invariant suggested by R. Berman and defined in [12]. Using the valuative criterion for K-stability of K. Fujita and C. Li [24, 36], it was shown that the invariant characterizes certain K-stability notions.

THEOREM 1.1 ([25, 5]). – *Let X be a $\mathbb{Q}$-Fano variety.*

(1) *X is uniformly K-stable iff $\delta(X) > 1$.*

(2) *X is K-semistable iff $\delta(X) \geq 1$.*

In light of the previous statement, Theorem A is a consequence of the following result.

THEOREM B. – *Let $\pi : X \to T$ be a projective flat family of varieties and L a π-ample Cartier divisor on X. Assume T is normal, X_t is a klt variety for all $t \in T$, and $K_{X/T}$ is $\mathbb{Q}$-Cartier. Then, the two functions $T \to \mathbb{R}$ defined by*

$$T \ni t \mapsto \delta(X_{\bar{t}}; L_{\bar{t}}) \quad \text{and} \quad T \ni t \mapsto \alpha(X_{\bar{t}}; L_{\bar{t}})$$

are lower semicontinuous.

In the above theorem, $(X_{\bar{t}}; L_{\bar{t}})$ denotes the restriction of (X, L) to the geometric fiber over t. As explained in [18, Remark 4.15], the above result would not hold with "$\delta(X_{\bar{t}}; L_{\bar{t}})$" replaced by "$\delta(X_t; L_t)$."

Let us note the main limitation of Theorem B. While the statement implies that $\{t \in T \mid \delta(X_{\bar{t}}; L_{\bar{t}}) > a\}$ is open for each $a \in \mathbb{R}_{\geq 0}$, it does not imply the fact that $t \mapsto \delta(X_{\bar{t}}; L_{\bar{t}})$ takes finitely many values. Hence, we are unable to prove $\{t \in T \mid \delta(X_{\bar{t}}; L_{\bar{t}}) \geq a\}$ is open and cannot verify the openness of K-semistability in families of $\mathbb{Q}$-Fano varieties. The openness of K-semistability is an immediate consequence of Theorem B and the following conjecture (see [6, Conjecture 2] for a local analogue).