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TOWARD EFFECTIVE LIOUVILLIAN INTEGRATION

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ABSTRACT. – We prove that foliations on the projective plane admitting a Liouvillian first integral but not admitting a rational first integral always have invariant algebraic curves of degree bounded by a function of the degree of the foliation. We establish, for the same class of foliations, the existence of a bound for the degree of the simplest integrating factor depending only on the degree of the foliation and on the nature of its singularities. We also prove the existence of invariant algebraic curves of small degree for foliations with rational first integral and intermediate Kodaira dimension.

RÉSUMÉ. – Vers l'intégration liouvillienne effective Nous montrons que les feuilletages du plan projectif qui admettent une intégrale première liouvillienne mais pas d'intégrale première rationnelle possèdent une courbe algébrique invariante dont le degré est majoré en fonction de celui du feuilletage. Pour la même classe de feuilletages, nous établissons une borne pour le degré du facteur intégrant le plus simple ; cette borne ne dépend que du degré du feuilletage et de la nature de ses singularités. Nous montrons également l'existence de courbes algébriques invariantes de petit degré pour les feuilletages à intégrale première rationnelle de dimension de Kodaira intermédiaire.

1. Introduction

This paper draws motivation from an ancient question studied by Poincaré, Autonne, Painlevé and others: Is it possible to decide if all the orbits of a polynomial vector field on the complex affine plane are algebraic?

Poincaré observes [36] that in order to provide a positive answer to the above question it suffices to bound the degree of the general orbit. Even if Poincaré is not explicit on which parameters the bound should depend on, in general, examples as simple as linear vector fields on \mathbb{C}^2 show that such a bound must depend on combinatorial data attached to the singularities of the vector field like their resolution process and quotients of eigenvalues of

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the resulting foliation. Soon after the appearance of Poincaré's paper, Painlevé writes in [29, pp. 216–217] the following paragraph.

J'ajoute qu'on ne peut espérer résoudre d'un coup qui consiste à limiter n . L'énoncé vers lequel il faut tendre doit avoir la forme suivante: "On sait reconnaître si l'intégrale d'une équation $F(y', y, x) = 0$ donnée est algébrique ou ramener l'équation aux quadratures." Dans ce dernier cas, la question reviendrait à reconnaître si une certaine intégrale abélienne (de première ou de troisième espèce) n'a que deux ou une périodes.⁽¹⁾

Painlevé suggests that one should first ask whether or not a given polynomial vector field admits a first integral "expressed through quadratures"; and only then, having this special first integral at hand, decide whether or not the leaves are algebraic. To put things in perspective it is useful to notice that the strategy to deal with the analogous problem for linear differential equations with rational coefficients is in accordance with Painlevé's suggestion cf. [3].

The vague terminology first integral "expressed through quadratures" can be formalized in several distinct ways. One possible interpretation is that one should look for first integrals belonging to a Liouvillian extension of the differential field $(\mathbb{C}(x, y), \{\partial_x, \partial_y\})$. For a precise definition and thorough discussion of this concept we refer to [39] and [14]. An important fact is Singer's theorem [39] that asserts that Liouvillian integrable foliations of the plane are *transversely affine* foliations. We recall that a foliation defined by a rational 1-form ω is transversely affine if ω admits an integrating factor, i.e., a closed rational 1-form η such that $d\omega = \omega \wedge \eta$. The class of transversely affine foliations includes the class of virtually transversely additive foliations. These are foliations which, after pull-back by a generically finite rational map, are defined by a closed rational 1-form.

More recently, it came to light a family of examples [22] showing the impossibility of giving bounds for the degree of a general algebraic orbit depending only on the analytical type of the singularities of the foliation/vector field. They consist of one parameter families of holomorphic foliations with fixed analytical type of singularities, such that the general foliation has only finitely many algebraic leaves and, in contrast, for a dense set of the parameter space the corresponding foliations have algebraic general leaf, of unbounded degree. These families of examples highlight the difficulties of Poincaré's original problem, and at the same time provide evidence for the effectiveness of the approach suggested by Painlevé as, in each family, all the members share a common integrating factor.

The reader not acquainted to the theory of holomorphic foliations on projective surfaces is invited to take a look at the first chapters of the reference textbook [9], see also [8]. Some important characters are the canonical bundle $K\mathcal{F}$ of the foliation and its Iitaka-Kodaira dimension [20] (or D -dimension) $\text{kod}(\mathcal{F})$.

⁽¹⁾ This could be translated as follows. *I add that we cannot hope to solve with a blow which consists in limiting n . The statement towards which it is necessary to tend must have the following form: "We can recognize if the integral of a given equation $F(y', y, x) = 0$ is algebraic or reduce the equation to quadratures." In the latter case, the question would amount to recognizing whether a certain abelian integral (of the first or third kind) has only two or one periods.*

1.1. Invariant algebraic curves of small degree

Our main results provide further evidence in favor of Painlevé's strategy. The first one shows that foliations on the projective plane admitting a Liouvillian first integral, but which do not admit a rational first integral, always possess an invariant algebraic curve of comparatively small degree.

THEOREM A. – *Let \mathcal{F} be a foliation of degree $d \geq 2$ on the projective plane \mathbb{P}^2 . Assume that \mathcal{F} admits a transverse affine structure but does not admit a rational first integral. Then \mathcal{F} admits an algebraic invariant curve of degree at most $12(d - 1)$. Moreover, if \mathcal{F} is not virtually transversely additive then there exists such a curve of degree at most $6(d - 1)$.*

The proof of Theorem A relies on recent results on the structure of transversely affine foliations, [15] and [24], that establish that a transversely affine foliation on a projective surface is either the pull-back under a rational map of a Riccati foliation; or is birationally equivalent to a finite quotient of a foliation defined by a closed rational 1-form. In both cases, our strategy consists in looking for sections of powers of the canonical bundle of the foliation vanishing along invariant algebraic curves. To achieve this in the case of pull-back of Riccati foliations we explore the description of the positive part of the Zariski decomposition of the canonical bundle of foliations of Kodaira dimension one due to McQuillan, see [26] and [8], in order to prove that, for some $k \leq 6$, $|K\mathcal{F}^{\otimes k}|$ defines a map to a curve which will contain the sought curves in its fibers. The case of finite quotients of foliations defined by closed rational 1-forms is trickier and makes use of the nonexistence of a rational first integral to guarantee the existence of a non-trivial representation of the fundamental group of the complement of the polar divisor of the transverse affine structure. We first deal with foliations defined by closed rational 1-forms. In this case, the non-trivial representation of the fundamental group allows us to produce logarithmic 1-forms generically transverse to the foliation and tangent to the zero divisor of the closed rational 1-form defining it. In the general case, one is asked to understand cyclic quotients of foliations defined by closed rational 1-forms. The proof goes on by studying the action of the relevant cyclic group on the space of symmetric logarithmic differentials and showing the existence of an invariant symmetric logarithmic differential of degree ≤ 12 tangent to the sought curves.

Of course, it would be highly desirable to have a similar result for foliations admitting a rational first integral. Unfortunately, our method to prove Theorem A exploits extensively the nonexistence of rational first integrals. Nevertheless, its use can be avoided in the case of foliations of Kodaira dimension zero or one. As a consequence we obtain a similar result for foliations admitting a rational first integral (also known as algebraically integrable foliations) such that the underlying fibration is isotrivial with fibers of genus $g \geq 2$; or the underlying fibration has elliptic fibers.

THEOREM B. – *Let \mathcal{F} be a foliation on \mathbb{P}^2 admitting a rational first integral and of intermediate Kodaira dimension, i.e., $\text{kod}(\mathcal{F}) \in \{0, 1\}$. Then one of the following situations occurs.*

1. *The foliation \mathcal{F} is birationally equivalent to an isotrivial fibration of genus 1 and \mathcal{F} admits an invariant algebraic curve of degree at most $6(d - 1)$.*
2. *The foliation \mathcal{F} is birationally equivalent to a non-isotrivial fibration of genus 1 and any irreducible algebraic curve invariant by \mathcal{F} has degree at most $12(d - 1)$.*