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STRATIFIED SURGERY AND K-THEORY INVARIANTS OF THE SIGNATURE OPERATOR

BY PIERRE ALBIN AND PAOLO PIAZZA

ABSTRACT. – In the works of Higson-Roe the fundamental role of the signature as a homotopy and bordism invariant for oriented manifolds is the starting point for an investigation of the relationships between analytic and topological invariants of smooth orientable manifolds. The signature and related K-theory invariants, primary and secondary, are used to define a natural transformation between the (Browder-Novikov-Sullivan-Wall) surgery exact sequence and a long exact sequence of C^*-algebra K-theory groups.

In recent years the primary signature invariants have been extended from closed oriented manifolds to a class of stratified spaces known as L-spaces or Cheeger spaces. In this paper we show that secondary invariants, such as the ρ-class, also extend from closed manifolds to Cheeger spaces. We give a rigorous account of a surgery exact sequence for stratified spaces originally introduced by Browder-Quinn and obtain a natural transformation analogous to that of Higson-Roe. We also discuss geometric applications.

RÉSUMÉ. – Dans les travaux de Higson-Roe le rôle fondamental de la signature comme invariant par homotopie et par bordisme de variétés orientées est le point de départ des recherches sur les liens entre les invariants analytiques et topologiques des variétés régulières orientées. La signature et certains invariants de K-théorie associés, primaires et secondaires, définissent une transformation naturelle entre la suite exacte de chirurgie de Browder-Novikov-Sullivan-Wall et une suite exacte longue des groupes de K-théorie pour des algèbres C^*.

Dans les dernières années l'étude des invariants de signature primaires des variétés orientées a été étendue à une classe d'espaces stratifiés connue sous le nom de L-espaces ou espaces de Cheeger. Dans ce papier, nous démontrons que les invariants secondaires, tels que la classe ρ, peuvent être étendus aux espaces de Cheeger. Nous traitons rigoureusement une suite exacte de chirurgie pour espaces stratifiés introduite originalement par Browder-Quinn et nous obtenons une transformation naturelle analogue à celle de Higson-Roe. Nous discutons aussi des applications géométriques.

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1. Introduction

The discovery by Milnor of smooth manifolds that are homemorphic to $\mathbb{S}^7$ but not diffeomorphic to it, a milestone of modern mathematics, gave rise to the development of methods for classifying smooth manifolds within a given homotopy class. (The undecidability of the word problem makes an unrestricted classification impossible.) The fundamental object to be studied in this context is the *structure set* $\mathrm{S}(X)$ associated to a smooth compact manifold X. The set $\mathrm{S}(X)$ is defined as the quotient of the set of triples $(M \xrightarrow{h} X)$, with M a smooth compact manifold and h an homotopy equivalence, modulo h-cobordism: $(M_0 \xrightarrow{h_0} X)$ is h-cobordant to $(M_1 \xrightarrow{h_1} X)$ if there exists a bordism $F : W \to X \times [0,1]$ with $\partial W = M_0 \sqcup M_1$, F restricting to f_j on M_j and F a homotopy equivalence. Notice that $\mathrm{S}(X)$ is a pointed set, with $[X \xrightarrow{\mathrm{Id}} X]$ as a distinguished point. It is in general very difficult to compute explicitly the structure set associated to X, a notable exception being the structure set of the spheres, $\mathrm{S}(\mathbb{S}^n)$. In this particular case, $\mathrm{S}(\mathbb{S}^n)$ can be identified with Θ_n, the Kervaire-Milnor group of h-cobordism classes of oriented homotopy n-spheres [33, 51]. Θ_n is a finite Abelian group, of cardinality 1 for $n \leq 6$, and, for example, cardinality 28 for $n = 7$. (The structure set of other simple spaces such as complex projective spaces, tori, and lens spaces are also known [57, Part 3].) In general there is no group structure on $\mathrm{S}(X)$ [(1)].

Even if an explicit computation is often out of reach, one would like to determine, for example, the cardinality of $\mathrm{S}(X)$, in particular whether it is greater than one, finite or infinite. A smooth manifold with $|\,\mathrm{S}(X)| = 1$ is said to be *rigid* and so $\mathrm{S}(X)$ is a measure of the non-rigidity of X.

A powerful method to get information about the structure set is provided by the surgery exact sequence of Browder, Novikov, Sullivan, and Wall which, roughly speaking, relates the structure set $\mathrm{S}(X)$ with the set $\mathrm{N}(X)$ of degree one maps preserving normal bundle information, known as 'normal invariants', (also with a bordism equivalence relation) and an algebraically defined L-group depending only on $\Gamma = \pi_1 X$, the fundamental group of X,

$$\cdots \longrightarrow \mathrm{L}_{m+1}(\mathbb{Z}\Gamma) \dashrightarrow \mathrm{S}(X) \longrightarrow \mathrm{N}(X) \longrightarrow \mathrm{L}_m(\mathbb{Z}\Gamma)\ . \tag{1.1}$$

with $m = \dim M$. (See below and, e.g., [57, 51, 36, 19] for more on the surgery exact sequence.)

In a series of papers Higson and Roe [27, 28, 29] established the remarkable result that there are natural maps out of the surgery sequence (1.1), into a long exact sequence of K-theory groups of certain C^*-algebras and that these maps make the resulting diagram commute. The C^*-algebras in question are $C^*(X_\Gamma)^\Gamma$ and $D^*(X_\Gamma)^\Gamma$, obtained as the closures of the Γ-equivariant operators on the universal cover X_Γ of X that satisfy a finite propagation property and, in addition, are respectively 'locally compact' or 'pseudolocal'. The former C^*-algebra is an ideal in the latter so we have a short exact sequence

$$0 \to C^*(X_\Gamma)^\Gamma \to D^*(X_\Gamma)^\Gamma \to D^*(X_\Gamma)^\Gamma / C^*(X_\Gamma)^\Gamma \to 0,$$

(1) The analogous set in the topological category, $\mathrm{S}^{\mathrm{top}}(X)$, X a topological manifold, can be given a group structure through the Siebenmann periodicity theorem, see for example [15].

which gives rise to a long exact sequence in K-theory known as the *analytic surgery sequence* of Higson and Roe. Making use of the canonical isomorphisms

$$K_{*+1}(D^*(X_\Gamma)^\Gamma / C^*(X_\Gamma)^\Gamma) = K_*(X) \quad \text{and} \quad K_*(C^*(X_\Gamma)^\Gamma) = K_*(C_r^*\Gamma),$$

with the K-homology of X and the K-theory of the reduced C^*-algebra of Γ, the long exact sequence reads

$$\cdots \to K_{m+1}(C_r^*\Gamma) \to K_{m+1}(D^*(X_\Gamma)^\Gamma) \to K_m(X) \to K_m(C_r^*\Gamma) \to \cdots . \tag{1.2}$$

The result of Higson and Roe is thus a commutative diagram of long exact sequences

$$\begin{array}{ccccccc} \mathrm{L}_{m+1}(\mathbb{Z}\Gamma) & \dashrightarrow & \mathrm{S}(X) & \longrightarrow & \mathrm{N}(X) & \longrightarrow & \mathrm{L}_m(\mathbb{Z}\Gamma) \\ \downarrow \gamma & & \downarrow \rho & & \downarrow \beta & & \downarrow \gamma \\ K_{m+1}(C_r^*\Gamma)[\frac{1}{2}] & \longrightarrow & K_{m+1}(D^*(X_\Gamma)^\Gamma)[\frac{1}{2}] & \longrightarrow & K_m(X)[\frac{1}{2}] & \longrightarrow & K_m(C_r^*\Gamma)[\frac{1}{2}], \end{array} \tag{1.3}$$

where we use the short-hand $A[\frac{1}{2}]$ to indicate $A \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{2}]$ whenever A is an Abelian group. These maps were recast by the second author and Schick [46] in a more index-theoretic light, using in a crucial way properties of the signature operator on Galois Γ-coverings. In particular, the homomorphism γ is shown to be realized by an Atiyah-Patodi-Singer index map. The approach by Piazza and Schick also allowed for a treatment of the Stolz surgery sequence for positive scalar curvature metrics and its mapping to the Higson-Roe surgery sequence using the spin-Dirac operator:

(1.4)

$$\begin{array}{ccccccc} R^{\mathrm{spin}}_{m+1}(B\Gamma) & \dashrightarrow & \mathrm{Pos}^{\mathrm{spin}}_m(B\Gamma) & \longrightarrow & \Omega^{\mathrm{spin}}_m(B\Gamma) & \longrightarrow & R^{\mathrm{spin}}_{m+1}(B\Gamma) \\ \downarrow g & & \downarrow \rho & & \downarrow b & & \downarrow g \\ K_{m+1}(C_r^*\Gamma)[\frac{1}{2}] & \longrightarrow & K_{m+1}(D^*(E\Gamma)^\Gamma)[\frac{1}{2}] & \longrightarrow & K_m(B\Gamma)[\frac{1}{2}] & \longrightarrow & K_m(C_r^*\Gamma)[\frac{1}{2}], \end{array}$$

see [45]. For alternative treatments see also [63, 66, 67, 68, 65]. While the vertical maps in these diagrams are in general not known to be injective or surjective (though see [64, Corollary 1.3] where this is related to the Baum-Connes conjecture), it is still possible to get interesting geometric applications from the interplay between the geometric sequence upstairs and the analytic sequence downstairs. This is true both for (1.3) and (1.4). See for example, [29, 17, 56, 66, 61, 60, 65, 10].

In this paper we generalize (1.3) to the setting of stratified spaces.

THEOREM 1.1. – *Every m-dimensional, oriented, smoothly stratified Cheeger space, $\widehat{X}$, with fundamental group Γ, gives rise to a commutative diagram*

$$\begin{array}{ccccccc} \mathrm{L}_{\mathrm{BQ},d_{\widehat{X}\times I}}(\widehat{X}\times I) & \dashrightarrow & \mathrm{S}_{\mathrm{BQ}}(\widehat{X}) & \longrightarrow & \mathrm{N}_{\mathrm{BQ}}(\widehat{X}) & \longrightarrow & \mathrm{L}_{\mathrm{BQ},d_{\widehat{X}}}(\widehat{X}) \\ \downarrow \mathrm{Ind}_{\mathrm{APS}} & & \downarrow \rho & & \downarrow \beta & & \downarrow \mathrm{Ind}_{\mathrm{APS}} \\ K_{m+1}(C_r^*\Gamma)[\frac{1}{2}] & \longrightarrow & K_{m+1}(D^*(\widehat{X}_\Gamma)^\Gamma)[\frac{1}{2}] & \longrightarrow & K_m(\widehat{X})[\frac{1}{2}] & \longrightarrow & K_m(C_r^*\Gamma)[\frac{1}{2}] \end{array} \tag{1.5}$$