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RECONSTRUCTING A VARIETY FROM ITS TOPOLOGY
[after Kollár, building on earlier work of Lieblich and Olsson]

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INTRODUCTION

The underlying Zariski topological space $|X|$ of an algebraic variety or, more generally, a scheme X tends to have few open subsets in comparison to topologies that underlie structures appearing in differential geometry or geometric topology. Thus, intuitively, $|X|$ is a weak invariant of X, and this intuition is confirmed by low-dimensional examples: for an algebraic curve C, the proper closed subsets of $|C|$ are the finite subsets of closed points, so $|C|$ does not see much beyond the cardinality of the algebraic closure of the base field. A more surprising example was constructed by Wiegand and Krauter (1981, Cor. 1): for primes p and p', there is a homeomorphism

$$|\mathbb{P}^2_{\overline{\mathbb{F}}_p}| \simeq |\mathbb{P}^2_{\overline{\mathbb{F}}_{p'}}|.$$

Topological spaces that underlie schemes (resp., affine schemes) were, in fact, completely classified by Hochster (1969, Thm. 9): they are the locally spectral (resp., the spectral) topological spaces. We recall that a topological space T is *spectral* if

— it is quasi-compact and quasi-separated;

— it is *sober*: each irreducible closed $T' \subset T$ is the closure $\overline{\{t\}}$ of a unique $t \in T'$;

— the quasi-compact open subsets form a base of the topology of T.

A topological space T is *locally spectral* if it has an open cover by spectral spaces. The topological space $|X|$ of a quasi-compact and quasi-separated scheme X is spectral, so Hochster's result implies that, somewhat surprisingly, $|X|$ also underlies some affine scheme. For instance, for any field k and any $n \geq 0$, the topological space $|\mathbb{P}^n_k|$ also underlies an affine scheme (which, of course, need not be a variety over a field).

Due to the above, the recent result of Kollár (2020), which is the focus of this report, came as a surprise: a projective, irreducible, normal variety X over $\mathbb{C}$ of dimension ≥ 4 is uniquely determined by its topological space $|X|$, see Theorem 1.1 below. A resulting general expectation in this direction is captured by the following conjecture of Kollár.

Conjecture 0.1 (Kollár, 2020, Conjecture 3). — *For seminormal, geometrically irreducible varieties X and X' over fields k and k', respectively, with $\operatorname{char} k = 0$ and $\dim X \geq 2$, every homeomorphism $|X| \xrightarrow{\sim} |X'|$ underlies a scheme isomorphism $X \xrightarrow{\sim} X'$.*

1. RECONSTRUCTION OF PROJECTIVE VARIETIES

The following result of Kollár builds on previous work of Lieblich and Olsson and fully resolves Conjecture 0.1 for projective, normal varieties of dimension ≥ 4 in characteristic 0. In fact, it forms the foundation of credibility for a conjecture of this sort.

Theorem 1.1 (Kollár, 2020, Theorem 2). — *For normal, geometrically integral, projective varieties X and X' over fields k and k', respectively, such that either*

1) $\dim X \geq 4$ *and both k and k' are of characteristic* 0; *or*

2) $\dim X \geq 3$ *and both k and k' are finitely generated field extensions of* $\mathbb{Q}$;

every homeomorphism $|X| \xrightarrow{\sim} |X'|$ underlies a scheme isomorphism $X \xrightarrow{\sim} X'$.

Remark 1.2. Since X and X' are proper and geometrically integral, we have isomorphisms $\Gamma(X, \mathscr{O}_X) \cong k$ and $\Gamma(X', \mathscr{O}_{X'}) \cong k'$, so a scheme isomorphism $X \xrightarrow{\sim} X'$ amounts to a field isomorphism $\iota \colon k \xrightarrow{\sim} k'$ and an isomorphism of varieties $X \otimes_{k,\iota} k' \xrightarrow{\sim} X'$.

We will focus on case 1) because it already contains most of the main ideas while avoiding further technicalities of 2) that largely concern the Hilbert irreducibility theorem. Roughly, the proof is based on studying Weil divisors D on a normal X: such D are determined by $|X|$ alone because they may be viewed as formal $\mathbb{Z}$-linear combinations of the points of codimension 1 (for instance, a reduced effective divisor $D \subset X$ is the closure of a finite set of codimension 1 points in X). We will let

$$\operatorname{Div}(X) := \bigoplus\nolimits_{x \in X^{(1)}} \mathbb{Z} \quad \text{and} \quad \operatorname{Eff}(X) := \bigoplus\nolimits_{x \in X^{(1)}} \mathbb{Z}_{\geq 0}$$

denote the group of all divisors (resp., the monoid of all effective divisors) on X.

It is not clear if notions such as ampleness or linear equivalence of divisors are determined by $|X|$ alone, and the crux of the argument is in exhibiting hypotheses under which they are. The ability to topologically recognize linear equivalence eventually reduces the reconstruction problem to a combinatorial recognition theorem for projective spaces in terms of incidence of their lines and points (von Staudt's fundamental theorem of projective geometry).

A divisor D on X is *ample* if some multiple nD with $n > 0$ is a Cartier divisor whose associated line bundle $\mathscr{O}(nD)$ is ample. We let $\sim$ denote linear equivalence of divisors and say that divisors D_1 and D_2 on X are *linearly similar*, denoted by $D_1 \sim_{\mathrm{s}} D_2$, if $n_1 D_1 \sim n_2 D_2$ for some nonzero integers n_1 and n_2. If this holds with $n_1 = n_2$, then we say that D_1 and D_2 are $\mathbb{Q}$*-linearly equivalent*, denoted by $D_1 \sim_{\mathbb{Q}} D_2$. When we speak of reduced (resp., irreducible) divisors, we implicitly assume that they are also effective (resp., effective and reduced). With these definitions, the overall proof of

Theorem 1.1 proceeds in the following stages, which successively reconstruct more and more of the structure of X from the topological space $|X|$, and which will be discussed individually in the indicated sections:

$$\begin{aligned} |X| &\overset{\S2}{\rightsquigarrow} \big(|X|, \sim_{\mathrm{s}} \text{ of irreducible ample divisors}\big) \\ &\overset{\S3\text{–}\S5}{\rightsquigarrow} \big(|X|, \sim \text{ of effective divisors}\big) \\ &\overset{\S6}{\rightsquigarrow} X. \end{aligned}$$

The last step, namely, the determination of a normal, geometrically integral, projective variety X of dimension ≥ 2 over an infinite field from its underlying topological space $|X|$ equipped with the relation of linear equivalence between effective divisors on $|X|$ is due to Lieblich and Olsson (2019).

The initial results of Lieblich and Olsson (2019), although already sufficient for Theorem 1.1 above, have been sharpened and expanded in Kollár, Lieblich, Olsson, and Sawin (2020).

2. RECOVERING LINEAR SIMILARITY OF AMPLE DIVISORS

Notation. *In this section, we let X be a normal, geometrically integral, projective variety over a field k of characteristic* 0.

The first stage of the proof of Theorem 1.1 is the reconstruction of linear similarity of irreducible ample divisors from the topological space $|X|$ alone. This requires, in particular, to be able to topologically recognize ampleness of irreducible divisors, which rests crucially on the following Lefschetz type theorem for the divisor class group.

Lemma 2.1 (Ravindra and Srinivas, 2006, Theorem 1). — *Suppose that* $\dim X \geq 3$ *and let $\mathscr{L}$ be an ample line bundle on X whose linear system $\Gamma(X, \mathscr{L})$ is basepoint free. For some nonempty Zariski open $U \subset \Gamma(X, \mathscr{L})$ and every effective divisor $D \subset X$ that corresponds to a k-point in U, the following restriction map is injective:*

$$\mathrm{Cl}(X) \hookrightarrow \mathrm{Cl}(D).$$

The cited result is sharper but only applies to the base change $X_{\bar{k}}$ to an algebraic closure $\bar{k}$. This suffices because $\mathrm{Cl}(X) \hookrightarrow \mathrm{Cl}(X_{\bar{k}})$: to see this last injectivity, note that for any divisor H on X that represents a class in the kernel, both $\mathscr{O}(H)$ and $\mathscr{O}(-H)$ have nonzero global sections, which, since X is projective, means that $H \sim 0$.

For proving Theorem 1.1 for varieties of dimension ≤ 4, one needs a refinement of Lemma 2.1 in which X is a surface (and D is a curve). This requires arithmetic inputs, notably a theorem of Néron (1952) on specialization of Picard groups. We refer to Kollár (2020, Thm. 74) for this refinement of Lemma 2.1. It would also be interesting to extend Lemma 2.1 to positive characteristic because this may be useful for establishing further cases of Conjecture 0.1. For instance, we could then weaken the assumption on k in this section: we could let it be a field that is not a subfield of any $\overline{\mathbb{F}}_p$.

The following is the promised topological criterion for ampleness.

Proposition 2.2 (Kollár, 2020, Lemma 67). — *Suppose that* $\dim X \geq 2$. *An irreducible divisor* $H \subset X$ *is ample if and only if for every effective divisor* $D \subset X$ *and distinct closed points* $x, x' \in X \setminus D$, *there is an effective divisor* $H' \subset X$ *with*

$$|H \cap D| = |H' \cap D| \quad \textit{and} \quad x \in H' \ \textit{but} \ x' \notin H'.$$

Sketch of proof. — To begin with the simpler direction, we assume that H is ample, replace it by a multiple to assume that H is Cartier with associated very ample line bundle $\mathscr{L}$, and fix a D and $x, x' \in X \setminus D$. By EGA III_1, 2.2.4, for some $n > 0$ there is an $s \in \Gamma(X, \mathscr{L}^{\otimes n})$ that vanishes at x, does not vanish at x', and is such that the vanishing locus of $s_{|D}$ is $H \cap D$. We can take H' to be the vanishing locus of s.

For the converse, we make a simplifying assumption that $\dim X \geq 3$ (for $\dim X = 2$ one needs a refinement of Lemma 2.1). To argue that H is ample, we will use Kleiman's criterion (Kleiman, 1966, Chap. III, Thm. 1 (i) ⇔ (iv) on p. 317), according to which it suffices to show that for all distinct closed points $x, x' \in X$, there exist an integer $n > 0$ and an effective divisor $\widetilde{H}$ such that $\widetilde{H} \sim nH$ and $x \in \widetilde{H}$ but $x' \notin \widetilde{H}$ (this will simultaneously prove that some nH is basepoint free, so is also Cartier, as we require of ample divisors). Since X is projective, Lemma 2.1 and the Bertini theorem applied to the irreducible components of $H_{\bar{k}}$ supply a normal effective divisor $D \subset X$ not containing x, x', or any generic point of H such that $H \cap D$ is irreducible and $\mathrm{Cl}(X) \hookrightarrow \mathrm{Cl}(D)$. By applying the assumption to this D, we find an effective divisor $H' \subset X$ with $|H \cap D| = |H' \cap D|$ and $x \in H'$ but $x' \notin H'$. Since $H \cap D$ is irreducible, this equality of topological spaces means that $nH_{|D} \sim n'H'_{|D}$ for some $n, n' > 0$. The injectivity of $\mathrm{Cl}(X) \hookrightarrow \mathrm{Cl}(D)$ then implies that $nH \sim n'H'$, and it remains to set $\widetilde{H} := n'H'$. □

Proposition 2.2 allows us to topologically recognize irreducible ample divisors on X. Granted this, the following proposition then expresses the linear similarity relation $\sim_{\mathrm{s}}$ between such divisors purely in terms of the topological space $|X|$.

Proposition 2.3 (Kollár, 2020, Lemma 68). — *Suppose that* $\dim X \geq 3$. *Irreducible divisors* $H_1, H_2 \subset X$ *with* H_1 *ample are linearly similar if and only if for any disjoint, irreducible, closed subsets* $Z_1, Z_2 \subset X$ *of dimension* ≥ 1 *there is an irreducible divisor* $H' \subset X$ *with*

$$|H_1 \cap Z_1| = |H' \cap Z_1| \quad \textit{and} \quad |H_2 \cap Z_2| = |H' \cap Z_2|.$$

Sketch of proof. — To begin with the simpler direction, we assume that $n_1 H_1 \sim n_2 H_2$ for some nonzero n_1, n_2 and fix Z_1, Z_2 as in the statement. The n_i must have the same sign: otherwise $\mathscr{O}(mH_1)$ and $\mathscr{O}(-mH_1)$ would have nonzero global sections for every large, sufficiently divisible $m > 0$. Thus, we may assume that $n_1, n_2 > 0$. After replacing n_1 and n_2 by nn_1 and nn_2 for a large $n > 0$, we then combine EGA III_1, 2.2.4 and the Bertini theorem (Jouanolou, 1983, 6.10) to find a global section of $\mathscr{O}(n_1 H_1) \simeq \mathscr{O}(n_2 H_2)$ whose vanishing locus is an irreducible ample divisor H' with the desired properties (and even such that the intersection of $H'_{\bar{k}}$ with every irreducible component of $X_{\bar{k}}$ is irreducible).

For the converse, we make a simplifying assumption that $\dim X \geq 5$ (to improve to $\dim X \geq 3$ one again needs a refinement of Lemma 2.1)—this time the assumption is more serious because the $\dim X \geq 5$ case does not suffice for Theorem 1.1. Letting H_1, H_2 be irreducible ample divisors as in the statement, we iteratively apply Lemma 2.1 (with the Bertini theorem) to build disjoint, irreducible, normal closed subschemes $Z_1, Z_2 \subset X$ that are complete intersections of dimension 2 such that $H_1 \cap Z_1 \subset Z_1$ and $H_2 \cap Z_2 \subset Z_2$ are irreducible divisors and the following restriction maps are injective:

$$\mathrm{Cl}(X) \hookrightarrow \mathrm{Cl}(Z_1) \quad \text{and} \quad \mathrm{Cl}(X) \hookrightarrow \mathrm{Cl}(Z_2).$$

Since the intersections $H_1 \cap Z_1$ and $H_2 \cap Z_2$ are irreducible, these injections and the displayed equalities involving H' ensure that $n_1 H_1 \sim n_1' H'$ and $n_2 H_2 \sim n_2' H'$ for some $n_i, n_i' > 0$. It then follows that $n_1 n_2' H_1 \sim n_1' n_2 H_2$, so that H_1 and H_2 are linearly similar, as desired. □

Propositions 2.2 and 2.3 jointly carry out the first reconstruction step promised in §1:

$$|X| \rightsquigarrow \big(|X|, \sim_{\mathrm{s}} \text{ of irreducible ample divisors}\big).$$

They also topologically determine complete intersection subvarieties as follows.

Corollary 2.4. — *Suppose that* $\dim X \geq 3$ *and let* $H \subset X$ *be an irreducible ample divisor. The topological space* $|X|$ *alone determines the collection of those closed subsets* $Z \subset |X|$ *that are* set-theoretic complete H-intersections, *i.e., for which there are irreducible divisors* $H_i \sim_{\mathrm{s}} H$ *for* $i = 1, \ldots, r$ *with* $r = \mathrm{codim}(Z, X)$ *such that*

$$Z = |H_1 \cap \ldots \cap H_r|.$$

Proof. — Propositions 2.2 and 2.3 imply that $|X|$ alone determines the property of H being ample, as well as the linear similarity relation $H_i \sim_{\mathrm{s}} H$. □

We will call such a closed subscheme $H_1 \cap \cdots \cap H_r \subset X$ a *complete H-intersection*. The requirement that the H_i be irreducible and only linearly similar (as opposed to linearly equivalent) to H makes this definition slightly nonstandard, but it is convenient because Propositions 2.2 and 2.3 only concern irreducible divisors. Any positive-dimensional complete H-intersection $H_1 \cap \ldots \cap H_r$ is automatically geometrically connected by the Lefschetz hyperplane theorem (SGA 2_{new}, XII, 3.5), and the same then also holds for set-theoretic complete H-intersections.

3. RECOVERING $\mathbb{Q}$-LINEAR EQUIVALENCE OF AMPLE DIVISORS

Notation. *In this section, we let X be a normal, geometrically integral, projective variety over field k of characteristic 0 and let $H \subset X$ be an irreducible ample divisor.*

To prepare for topological recognition of linear equivalence of divisors, for now we continue to restrict to irreducible ample divisors and show how to recognize $\mathbb{Q}$-linear equivalence between them. This refines the result presented in the previous section because $\mathbb{Q}$-linear equivalence $\sim_{\mathbb{Q}}$ is a finer relation than linear similarity $\sim_{\mathrm{s}}$. In addition, it involves techniques that will also be relevant later, such as topological