Astérisque
330, 2010, p. 1–11

BANACH ℓ-ADIC REPRESENTATIONS OF p-ADIC GROUPS

by

Marie-France Vigneras

Abstract. — Let $p \neq \ell$ be two distinct prime numbers, let F be a p-adic field and let E be an ℓ-adic field. We prove that the smooth part and the completion are inverse equivalences of categories between the category of admissible Banach unitary E-representations of $GL(n, F)$ and the category of admissible smooth E-representations of $GL(n, F)$ equipped with a commensurability class of lattices. We formulate the ℓ-adic local Langlands correspondence as a canonical bijection between the n-dimensional ℓ-adic representations of the absolute Galois group Gal_F and the topologically irreducible admissible Banach unitary ℓ-adic representations of $GL(n, F)$.

***Résumé* (Représentations ℓ-adiques de groupes p-adiques).** — Soient $p \neq \ell$ deux nombres premiers distincts, soit F un corps p-adique et soit E un corps ℓ-adique. Nous démontrons que la partie lisse et la complétion définissent des équivalences de catégories inverses l'une de l'autre entre la catégorie des représentations admissibles de Banach unitaires de $GL(n, F)$ sur E et la catégorie des représentations lisses admissibles de $GL(n, F)$ sur E munies d'une classe de commensurabilité de réseaux. Nous formulons la correspondance de Langlands locale ℓ-adique comme une bijection canonique entre les représentations ℓ-adiques de dimension n du groupe de Galois absolu Gal_F et les représentations topologiquement irréductibles admissibles de Banach unitaires ℓ-adiques de $GL(n, F)$.

1. Introduction

Let p be a prime number, let F be a finite extension of $\mathbf{Q}_p$ or a field of Laurent series $k((T))$ over a finite field k of characteristic p, let $\overline{F}$ be an algebraic closure of F and let n be an integer ≥ 1.

For any topological field C, the continuous representations of $GL(n, F)$ on topological vector spaces over C are interesting for their applications in arithmetic, geometry or physics, via the theory of L-functions associated to automorphic representations. When C varies, the theories of C-representations of $GL(n, F)$ present simultaneously

2010 Mathematics Subject Classification. — 11F70, 11F80, 11F85.
Key words and phrases. — Banach ℓ-adic representation, reductive p-adic group, admissible representation, smooth representation, lattice.

strong similarities and strong different features but the Langlands insight, when C is the complex field, to use the smooth complex representations of $\mathrm{Gal}_F = \mathrm{Gal}(\overline{F}/F)$ as a classifying scheme, seems to extend to other fields.

Why moving the coefficient field C? There are many reasons.

1) The representations of Gal_F appearing naturally are not smooth complex. In the étale cohomology of proper smooth algebraic varieties, they are continuous ℓ-adic representations on finite dimensional vector spaces V over finite extensions $E/\mathbf{Q}_\ell$, for a prime number ℓ. By a reduction of a stable O_E-lattice of V, they give smooth mod ℓ-representations over the residual field of E.

2) The local Langlands correspondence for $GL(n,F)$, over any algebraically closed field R of characteristic different from p, is a bijection

$$\pi \leftrightarrow (\rho, N)$$

between the equivalence classes of the smooth irreducible R-representations π of $GL(n,F)$ and of the pairs (ρ, N) where ρ is a n-dimensional smooth semi-simple R-representation of the Weil group W_F and N a nilpotent endomorphism of the space of ρ such that $\rho(w)N = N|w|\rho(w)$ where $|?|$ is the unramified R-character of W_F sending a geometric Frobenius to q, the order of the residual field of F.

Our purpose is to obtain a local Langlands correspondence for continuous ℓ-adic representations.

Theorem 1. — *Let ℓ be a prime number different from p. The ℓ-adic local Langlands correspondence for $GL(n,F)$ is a canonical bijection between the equivalence classes of*

a) *n-dimensional continuous ℓ-adic representations of Gal_F with a semi-simple action of the Frobenius,*

b) *topologically irreducible admissible Banach unitary ℓ-adic representations of $GL(n,F)$.*

This theorem[(1)] is motivated by the fascinating work and conjectures of Christophe Breuil on the p-adic local Langlands correspondence, where topologically irreducible admissible Banach unitary p-adic representations of $GL(2,\mathbf{Q}_p)$ appear naturally.

With the existing literature, one translates the local Langlands complex correspondence for $GL(n,F)$ into a canonical bijection between the isomorphism classes of a) and of

c) *Irreducible smooth $\overline{\mathbf{Q}}_\ell$-representations of $GL(n,F)$ with a stable lattice.*

Indeed, as is well known,

(i) The smooth complex local Langlands correspondence $LL(\rho,N)$ twisted by a suitable unramified character,

$$(\rho, N) \leftrightarrow LL(\rho, N) \otimes |\det ?|^{-(n-1)/2},$$

(1) Proved in a letter to Breuil in september 2003, and announced in the Emmy Noether lectures 2005 of Goettingen.

called the smooth complex local Hecke correspondence, is Aut $\mathbf{C}$-equivariant [H prop.6].

(ii) Transporting the correspondence (i) with an algebraic isomorphism $j : \mathbf{C} \simeq \overline{\mathbf{Q}}_\ell$, we obtain the smooth local Hecke $\overline{\mathbf{Q}}_\ell$-correspondence, which does not depend on the choice of the isomorphism j.

(iii) N disappears when one considers continuous $\overline{\mathbf{Q}}_\ell$-representations of W_F instead of smooth $\overline{\mathbf{Q}}_\ell$-representations. The pairs (ρ, N) are in bijection

$$(\rho, N) \leftrightarrow \sigma$$

with the n-dimensional ℓ-adic representations σ of W_F with a semi-simple action of the Frobenius. The reason is that the kernel of the natural morphism $t : I_F \to \mathbf{Z}_\ell$ is a profinite group prime to ℓ. There is a nilpotent endomorphism N of the space of σ such that $\sigma(?) = \exp(t(?)N)$ on a subgroup of finite index of I_F [**8**].

(iv) The n-dimensional ℓ-adic representation σ of W_F in (iii) extends by continuity to an ℓ-adic representation of Gal_F if and only if ρ has a bounded image (i.e. the values of determinants of the irreducible components of ρ are units) [**8**].

(v) ρ has a bounded image if and only if $\pi = LL(\rho, N)$ is integral [**10**, §1.4]; moreover all stable lattices in π are commensurable [**11**, Theorem 1].

Our task is to show that the completion with respect to a stable lattice gives a bijection between the isomorphism classes of b) and of c).

The beginning of the proof is valid for any locally profinite group G, with a countable fundamental system of neighborhoods of the unit, consisting of open profinite groups of pro-order not divisible by ℓ (Section 2). We prove (Theorem 2.12) that the completion and the smooth part induce equivalences of categories between the category $\mathscr{M}_\ell(G)^{\mathrm{adm}}$ of admissible smooth ℓ-adic representations of G equipped with a commensurability class of lattices, and the category $\mathscr{B}_\ell(G)^{\mathrm{adm}}$ of admissible Banach unitary *ℓ-adic* representations of G.

Then we consider the group of rational points G_F of any reductive connected group over a local non Archimedean field F of residual characteristic $p \neq \ell$ (Section 3). We prove (Theorem 3.6) that the completion and the smooth part induce equivalences of categories between the category $\mathrm{Mod}^{\mathrm{int,fl}}_{\overline{\mathbf{Q}}_\ell}(G_F)$ of integral smooth $\overline{\mathbf{Q}}_\ell$-representations of G_F of finite length and the category $\mathscr{B}_\ell(G_F)^{\mathrm{adm,fl}}$ of admissible Banach unitary *ℓ-adic* representations of topological finite length of G_F. We deduce the wanted bijection between the isomorphism classes of b) and c) by restricting to irreducible representations and choosing $G_F = GL(n, F)$.

A natural question was raised by the referee: Is a topologically irreducible Banach unitary *ℓ-adic* representation of G_F always admissible ? L. Clozel noticed that the examples of B. Diarra [**5**, th. 4] (van Rooj), give examples of topologically irreducible representations $V \in \mathscr{B}_E(GL(1, F))$ where any non zero intertwining operator is bijective, which are *not admissible.*

2.

2.1. The two categories. — Let $\ell \neq p$ be two distinct prime numbers, let $E/\mathbf{Q}_\ell$ be a finite extension of ring of integers O_E, of uniformizer p_E, and of residual field k_E, and let G be a topological group admitting a *countable* fundamental system of neighborhoods of the unit consisting of open *pro-ℓ'*-subgroups (profinite subgroups of pro-order *prime to* ℓ).

After having recalled some definitions and properties concerning the representations of the group G on E-vector spaces, we will introduce the two categories of representations $\mathscr{M}_E(G)$ and $\mathscr{B}_E(G)$ which will be compared in this paper.

Let Mod_E be the category of E-vector spaces and let $M \in \mathrm{Mod}_E$ non zero. A *line* in M is a subspace of dimension 1. A *lattice* L in M is a O_E-submodule of M which contains *no line* and contains a basis of M over E. Note that a quotient of a lattice may contain a line. When the dimension of M over E is *countable*, a lattice L in M is a *free* O_E-submodule of M generated by a basis of M over E [**9**, I Appendice C.5]. Two lattices L, L' in M are *commensurable* when there exists an element $a \in O_E$ such that $aL \subset L'$, $aL' \subset L$. We denote by $[L]$ the commensurablility class of L.

Remark 2.1. — *An O_E-submodule L of $M \in \mathrm{Mod}_E$ is a lattice in M if and only if any non zero element $m \in M$ satisfies the two conditions:*

a) *there exists an integer $n \in \mathbf{N}$ such that $\ell^n m$ belongs to L,*

b) *there exists an integer $n \in \mathbf{N}$ such that $\ell^{-n} m$ does not belong to L.*

Two lattices L, L' in M are commensurable if and only if there exists an integer $n \in \mathbf{N}$ such that $\ell^n L \subset L'$, $\ell^n L' \subset L$.

A representation (= a linear action) of G on M is called *admissible* when $\dim_E M^H < \infty$, for any open pro-ℓ'-subgroup H of G, where $M^H \in \mathrm{Mod}_E$ is the subspace of H-invariant vectors of M. The representation M is called *irreducible* when $M \neq 0$ and 0 and M are the only G-stable subspaces of M, *finitely generated* when M is a finitely generated EG-module, *of finite length* when there exists a finite G-stable filtration $0 \subset M_1 \subset \cdots \subset M_n = M$ with *irreducible quotients.* The length of the filtration and the isomorphism classes of the quotients, up to the order, do not depend on the choice of the filtration.

A *lattice* L in the *representation* of G on M will always be a *G-stable* lattice in M; the lattice will be called *finitely generated* when it is a finitely generated $O_E G$-module. A representation of G on M containing a lattice is called *integral* (we do not suppose that the lattice is O_E-free as in [**9**]). There exist finitely generated lattices in a finitely generated integral representation; they form a commensurability class, and any lattice contains a finitely generated lattice.

A *continuous* E-representation of G is a topological Hausdorff E-vector space M equipped with a continuous action of G, i.e. such that the map $(g, v) \to gv : G \times M \to M$ is continuous. It is called *topologically irreducible* when $M \neq 0$ and 0 and M are the only *closed* G-stable subspaces of M. It is called of *finite topological length* when there exists a finite filtration by G-stable *closed* subspaces $0 \subset M_1 \subset \cdots \subset M_n = M$ with *topologically irreducible quotients.*

The category $\mathscr{C}_E(G)$ of continuous representations of G on topological Hausdorff *complete* E-vector spaces with continuous G-equivariant E-linear morphisms, called *intertwinning operators*, contains the subcategory $\mathrm{Mod}_E(G)$ of smooth representations and the subcategory $\mathscr{B}_E(G)$ of Banach unitary representations, defined below. We indicate by the upper index *adm* or *fl* or adm, fl or int or int, fl the full subcategories representations which are admissible or of finite topological length or admissible and of finite topological length or integral or integral and of finite topological length. Example: $\mathscr{C}_E(G)^{adm}, \mathrm{Mod}_E(G)^{\mathrm{adm}}, \mathscr{B}_E(G)^{\mathrm{adm}}$ for admissible representations.

A representation of G on an E-vector space W is *smooth* when the stabilizer in G of any vector of W is open; this is simply a continuous representation of G on W when W is equipped with the discrete topology. The category $\mathrm{Mod}_E(G)$ of smooth E-representations of G, with morphisms the G-equivariant E-linear maps, is a full subcategory of $\mathscr{C}_E(G)$.

A *Banach unitary* E-representation V of G is a Hausdorff complete topological E-vector space with a topology given by a norm, equipped with a continuous action of G which respects the norm. A unit ball of V is $L = \{v \in V \ : ||v|| \leq 1\}$ for some norm $v \mapsto ||v||$ on V defining the topology [Sch I.3, III]; it is a lattice in V. The unit balls of two norms on V giving the same topology are commensurable.

An E-linear map $f : V_1 \to V_2$ between two Banach E-vector spaces V_1, V_2 is continuous if and only if there exists some non zero $a \in E$ such that $f(L_1) \subset af(L_2)$ for some unit balls L_1, L_2 of V_1, V_2 [Sch I.3.1]. The topology quotient topology on the image of f is the topology induced by V_2 if and only if $f(L_1)$ and $L_2 \cap f(V_1)$ are commensurable (this does not depend on the choice of the unit balls L_1, L_2). When f is continuous and bijective, the inverse of f is continuous [Sch I.8.7].

We will compare $\mathscr{B}_E(G)$ with the category $\mathscr{M}_E(G)$ of smooth E-representations W of G equipped with a commensurability class $[L]$ of lattices; a morphism $(W, [L]) \to (W', [L'])$ is a morphism $f : W \to W'$ in $\mathrm{Mod}_E(G)$ such that $f(L) \subset aL'$ for some $a \in E$. The pair $(W, [L])$ is called admissible or of finite length when W is admissible or of finite length, and $\mathscr{M}_E(G)^{\mathrm{adm}}$ or $\mathscr{M}_E(G)^{fl}$ is the full subcategory of admissible or of finite length pairs in $\mathscr{M}_E(G)$.

2.2. The two functors. — We introduce two natural functors in opposite directions between the categories $\mathscr{M}_E(G)$ and $\mathscr{B}_E(G)$.

There is the natural functor $\mathscr{C}_E(G) \to \mathrm{Mod}_E(G)$ sending $M \in \mathscr{C}_E(G)$ to its *smooth part*

$$M^\infty := \cup_H M^H,$$

for all open pro-ℓ'-subgroups H of G. When $V \in \mathscr{B}_E(G)$ is a Banach unitary representation of G, the smooth part $L^\infty = V^\infty \cap L$ of a unit ball L of V is a lattice of V^∞. Two unit balls of V are commensurable and their smooth parts are commensurable, hence $(V^\infty, [L^\infty]) \in \mathscr{M}_E(G)$ is well defined. A continuous morphism $f : V_1 \to V_2$ of Banach unitary E-representations of G with unit balls L_1, L_2, restricts to a morphism