

## THE SIGNATURE OPERATOR ON MANIFOLDS WITH A CONICAL SINGULAR STRATUM

by

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*Dedicated to Jean-Michel Bismut on the occasion of his 60<sup>th</sup> birthday*

**Abstract.** — We consider a Riemannian manifold,  $M$ , which can be compactified by adjoining a smooth compact oriented Riemannian manifold such that a neighbourhood of the singular stratum  $B$ , of codimension at least two, is given by a family of metric cones. Under the assumption that the middle cohomology of the cross-section vanishes, we show that there is a natural self-adjoint extension for the Dirac operator on forms with discrete spectrum, and we determine the condition of essential self-adjointness. We describe the boundary conditions analytically and construct a good parametrix which leads to the asymptotic expansion of a suitable resolvent trace as in our previous work. We also give a new proof of the local formula for the  $L^2$ -signature.

**Résumé (Opérateur de signature sur les variétés avec une strate singulière conique)**

Nous considérons une variété riemannienne  $M$ , qui peut être compactifiée en lui adjoignant une variété riemannienne  $C^\infty$  compacte orientée, telle qu'un voisinage de la strate singulière  $B$ , de codimension au moins deux, est donné par une famille de cônes métriques. Sous une hypothèse d'annulation de la cohomologie de la section du cône en dimension moitié, nous montrons qu'il existe une extension auto-adjointe naturelle de l'opérateur de Dirac agissant sur les formes qui est de spectre discret, et nous déterminons la condition sous laquelle l'opérateur de Dirac est essentiellement auto-adjoint. Nous décrivons les conditions de bord, et nous construisons une parametrix qui donne le développement asymptotique de la trace de la résolvante, comme dans un travail antérieur. Nous donnons aussi une preuve nouvelle de la formule locale pour la signature  $L^2$ .

### Introduction

In this article, we analyze the signature operator on an oriented Riemannian manifold  $(M, g)$ , of dimension  $m = 4k$ , with one compact singular stratum  $B$  of dimension

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$h$  (the “horizontal dimension”), such that  $m - h \geq 2$ . A neighbourhood of the singular set is given by

$$(0.1) \quad U := U_{\varepsilon_0} := (0, \varepsilon_0) \times N, \quad \varepsilon_0 \in (0, 1/2),$$

with an oriented compact Riemannian manifold  $N$  of dimension  $4k - 1$  and metric  $g^{TN}$ , and  $M$  decomposes as

$$(0.2) \quad M =: U_{\varepsilon_0} \cup M_{\varepsilon_0}$$

into points of distance at most and at least  $\varepsilon_0$  of the singular set, respectively. For  $\varepsilon \in (0, \varepsilon_0]$ , we use analogous notation and write  $U_\varepsilon, M_\varepsilon$ , with

$$M = U_\varepsilon \cup M_\varepsilon.$$

We assume that the orientation on  $M$  and  $N$  induce the boundary orientation on  $U$ , such that  $\{-\frac{\partial}{\partial t}, e_1, \dots, e_{m-1}\}$  is oriented on  $U$  if  $t \in (0, \varepsilon_0)$  and  $\{e_1, \dots, e_{m-1}\}$  is oriented on  $N$ . We assume in addition that the singularity is of the following special type. There is a fibration of oriented compact Riemannian manifolds,

$$(0.3) \quad \pi : Y \hookrightarrow N \rightarrow B,$$

with fibers  $Y_b = \pi^{-1}(b)$ ,  $b \in B$ , of dimension  $v := 4k - 1 - h \geq 1$  (the “vertical dimension”); in particular,  $B$  carries a metric  $g^{TB}$  such that  $\pi$  becomes a Riemannian submersion. Then the tangent bundle  $TN$  of  $N$  splits under  $g^{TN}$  into the *vertical* and the *horizontal* tangent bundle, consisting of the tangent vectors to the fibers and their orthogonal complement,

$$(0.4) \quad TN_p =: T_H N_p \oplus T_V N_p,$$

with induced metrics  $g^{T_H N}$  and  $g^{T_V N}$ ; the corresponding orthogonal projections in  $TN$  will be denoted by  $P_H$  and  $P_V$ , respectively. Next we assume that the metric  $g^{TU} := g^{TM}|_U$  takes the form

$$(0.5) \quad g^{TU} := dt^2 \oplus g^{T_H N} \oplus t^2 g^{T_V N},$$

which we will call a metric of *conic type*. Thus,  $M \cup B$  is a Riemannian pseudomanifold with one singular stratum of conic type.

The boundary of  $U$  is the Riemannian manifold

$$(0.6) \quad N_{\varepsilon_0} := (N, g_{\varepsilon_0}^{TN} := g^{T_H N} \oplus \varepsilon_0^2 g^{T_V N}).$$

The splitting of  $TN$  induces a splitting of the cotangent bundle,

$$T^*N =: T_H^*N \oplus T_V^*N,$$

into cotangent vectors annihilating  $T_V N$  and  $T_H N$ , respectively. This splitting induces a bigrading of the exterior algebra  $\Lambda T^*N$  which will be important for our analysis; we write

$$(0.7) \quad \begin{aligned} \Lambda T^*N &= \Lambda T_H^*N \otimes \Lambda T_V^*N \\ &= \bigoplus_{j=p+q} \Lambda^p T_H^*N \otimes \Lambda^q T_V^*N =: \bigoplus_{p,q} \Lambda^{p,q} T^*N. \end{aligned}$$

The smooth sections of  $\Lambda T^*N$  and  $\Lambda T_{H/V}^*N$  will be denoted  $\lambda(N)$  and  $\lambda_{H/V}(N)$ , respectively, with degree or bidegree noted with superscripts.

Our main object will be the canonical Dirac operator associated with  $\Lambda T^*M$ ,

$$(0.8) \quad D^\Lambda := D_M^\Lambda := d_M + d_M^\dagger,$$

with  $d_M =: d$  the exterior derivative on  $M$  and  $d^\dagger$  its formal adjoint with respect to the metric  $g^{TM}$ .

$D$  defined on forms with compact support, denoted by  $\lambda_c(M)$ , is symmetric in  $L^2(M, \Lambda T^*M) =: \lambda_{(2)}(M)$  but may not be essentially self-adjoint; we refer to the closure of this operator as  $D_{\min}^\Lambda =: D_{\min}$ , and  $d_{\min}, d_{\min}^\dagger$  are defined analogously.

A specific self-adjoint extension of this operator can be defined via the Hilbert complex given by the operator  $d_{\max}$  which arises from  $d^\dagger$  as

$$(0.9) \quad d_{\max} := (d_{\min}^\dagger)^*,$$

cf. [11, §3]; with a slight abuse of notation we denote this extension again by  $D = D^\Lambda = D_M^\Lambda$ , with domain  $\mathcal{D} = \text{dom } D$ . In general, there will be many more self-adjoint extensions but  $D$  is of interest since its kernel gives the  $L^2$ -cohomology of  $M$ . If  $D$  is a Fredholm operator we have to break its symmetry to obtain a nontrivial index, e. g. by an anticommuting *supersymmetry* i. e. a self-adjoint involution of  $\Lambda T^*M$ . We will use multiplication by the complex volume element,  $\tau_M$ , which splits

$$\Lambda T^*M =: \Lambda^+ T^*M \oplus \Lambda^- T^*M$$

into  $\pm 1$ -eigenbundles and analogously

$$\lambda(M) =: \lambda^+(M) \oplus \lambda^-(M),$$

with associated splitting  $\sigma = \sigma^+ + \sigma^-$  on the level of forms. If  $\tau_M$  maps  $\mathcal{D}$  to itself then we can define the *Signature Operator* of  $M$ , with domain  $\mathcal{D}^{\text{sign}} = \mathcal{D}^+ = \frac{1}{2}(I + \tau_M)\mathcal{D}$ , by

$$(0.10) \quad D_M^{\text{sign}} := D^{\text{sign}} := D_M^\Lambda |_{\mathcal{D}^+} : \mathcal{D}^+ \rightarrow \mathcal{D}^-.$$

We say that the *case of uniqueness* or the  $L^2$ -Stokes Theorem holds on  $M$  if

$$(0.11) \quad d_{\max} = d_{\min}.$$

In this case we have  $\tau(\mathcal{D}) \subset \mathcal{D}$ , and if  $D$  is also Fredholm, then so is  $D^{\text{sign}}$  and its index equals the  $L^2$ -signature of  $M$ ,

$$(0.12) \quad \text{ind } D^{\text{sign}} = \text{sign}_{(2)} M.$$

The above metric data define the crucial object in the analysis of the signature operator: the splitting of  $T^*N$  (induced by (0.4)) defines the “vertical de Rham operator”  $d_V$  (see (2.5)) and the metric  $g_1^{T_V N}$  defines the adjoint  $d_V^\dagger$ , such that we can form the operator (see (2.31))

$$(0.13) \quad A_V := (d_V + d_V^\dagger)\alpha + \nu.$$

Here  $\alpha$  is another supersymmetry on  $\Lambda T^*N$  and  $\nu$  is an endomorphism (which are defined in (2.19) and (2.13)), and  $A_V$  is a first order symmetric differential operator on  $C_c^1(N, \Lambda T^*N)$  which is fiberwise elliptic. Now  $M$  is called a *Witt space* if

$$(0.14) \quad H^{v/2}(Y) = 0.$$

We will see below (cf. Theorem 3.1) that (0.14) is essentially equivalent to the analytic condition

$$(0.15) \quad A_V \text{ is invertible,}$$

in the sense that the invertibility of  $A_V$  implies the Witt condition, whereas the Witt condition does not exclude the existence of zero eigenvalues but only of such which may be called inessential; indeed, they disappear under suitable rescalings of the fiber metric. *We will assume that  $M$  is a Witt space.*

Our results can then be summarized in the following theorems. We describe the Signature Operator on  $M$  by explicitly constructing its Green kernel which relates it to the symmetric operator  $\tilde{D}$  defined as the restriction of  $D_{\max}$  to the domain

$$(0.16) \quad \{\sigma \in \text{dom } D_{\max} : \|\sigma^+\|_{\lambda_{(2)}(N_t)} = O(t^{1/2-\varepsilon}) \text{ for every } \varepsilon > 0, \\ \|\sigma^-\|_{\lambda_{(2)}(N_t)} = O(t^{-1/2+\eta}) \text{ for some } \eta > 0, t \rightarrow 0\};$$

note that  $\tilde{D}$  anticommutes with  $\tau_M$  by construction.

**Theorem 0.1.** — *Let the Riemannian manifold  $(M, g^{TM})$ , of dimension  $m = 4k$ , be the top stratum of a Riemannian pseudomanifold,  $X$ , which is a Witt space with only one singular stratum  $B$  of conic type.*

1. *The operator  $\tilde{D}$  defined by (0.16) is self-adjoint and discrete and anticommutes with  $\tau_M$ .*
2. *If  $|A_V| \geq \frac{1}{2}$ , then  $D_{M, \min}^\Lambda$  is essentially self-adjoint.*
3. *The case of uniqueness holds for  $M$ .*
4.  *$D^{\text{sign}} = \tilde{D}^+$ .*

This theorem is well known in the case  $h = 0$ , cf. [15], [12], and part 2 and part 3 could also be deduced from Cheeger's work [15].

It is clear from part 4 of Theorem 0.1 that under the above conditions

$$(0.17) \quad \text{ind } \tilde{D}^+ = \text{ind } \tilde{D}^{\text{sign}} = \text{sign}_{(2)}M,$$

so it is natural to ask for a local formula analogous to Hirzebruch's Signature Theorem in the smooth case. Bismut and Cheeger [6, Thm. 5.7] have indicated the adiabatic construction of the homology  $L$ -class on the compact singular space associated with  $M$ , together with the corresponding  $L^2$ -index formula. A crucial role is played by the  $\eta$ -invariant,  $\eta(N, g^{TN})$ , of the Riemannian manifold  $(N, g^{TN})$ , as introduced by Atiyah, Patodi, and Singer in [1, Thm. (4.14)], and its adiabatic limit,

$$\tilde{\eta}(N, g^{TN}) := \lim_{\varepsilon \rightarrow 0} \eta(N, g_\varepsilon^{TN}).$$

The adiabatic limit was first introduced and computed by Witten [25], as a gravitational anomaly, in case of a one-dimensional base. Witten's formula was proved rigorously by Cheeger [16], and independently by Bismut and Freed [9], [10]. The computation of the adiabatic limit for arbitrary dimensions and invertible fiber operators was given by Bismut and Cheeger [6, 7], who introduced the form  $\tilde{\eta} = \tilde{\eta}(\pi, g^{TM}) \in \lambda(B)$  generalizing the  $\eta$ -invariant; the case of the signature operator was treated by Dai [18, Thm.0.3] who further introduced the  $\tau$ -invariant associated to the Leray spectral sequence of the fibration (0.3). There has been done considerable work recently on the computation of  $L^2$ -cohomology groups of spaces which can be compactified as pseudo-manifolds of the type we consider here, cf. [19], [20], [21], and [17]. These calculations lead to topological formulas for  $\text{sign}_{(2)}M$ , see [17, Cor.1.2] for Witt spaces and its extension in [21]. Combining these topological formulas with Dai's result quoted above gives the following local signature formula which was stated for even dimensional base spaces in [8, Thm.5.7]; in its formulation, we denote by  $D_B^{\Lambda \otimes \mathcal{H}(Y)}$  the Dirac operator  $D_B^\Lambda$  twisted by the bundle of fiber harmonic forms.

**Theorem 0.2.** — *We have*

$$\text{ind } D^{\text{sign}} = \lim_{\varepsilon \rightarrow 0} \int_M L(\nabla^{TM}) - \int_B L(TB, \nabla^{TB}) \wedge \tilde{\eta} - \frac{1}{2} \eta(D_B^{\Lambda \otimes \mathcal{H}(Y)}).$$

We give here an analytic proof of [17, Cor.1.2] in the general case which should be applicable to more general situations; in combination with the results of Atiyah, Patodi, and Singer and Dai's computation, it yields the theorem. The parametrix construction which we give in this paper should, in principle, also lead directly to the local index formula but, so far, we have been unable to overcome the technical difficulties involved.

We also have considered the resolvent trace expansion. We have a proof of the following result, but its presentation would lengthen the paper unduly; we hope to include it in a more general result at some future time.

**Theorem 0.3.** — 1. *For  $\mu \in \mathbb{R} \setminus \{0\}$  and  $p > m$ , the resolvent  $(D - i\mu)^{-1}$  is in the Schatten-von Neumann class of order  $p$  in  $L^2(M, \Lambda T^*M)$ .*  
 2. *For  $z \in \mathbb{R}$  and  $l > m/2$ , we have the asymptotic expansion*

$$\text{tr}[D^2 + z^2]^{-l} \sim_{z \rightarrow \infty} z^{m-2l} \sum_{j \geq 0} a_j z^{-j} + \sum_{j \geq 2l-h} b_j z^{-j} \log z.$$

The plan of the article is as follows. In Section 1, we deal with general Dirac operators and derive some decomposition theorems which are induced by a fibration of the form (0.3) and are needed later on. These results are known for spin Dirac operators, see [5, pp. 56, 59].

In Section 2, we represent the signature operator  $D^{\text{sign}}$  on  $U$  in the form

$$D_M^{\text{sign}} \simeq \frac{\partial}{\partial t} + A_H(t) + t^{-1} A_V,$$