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AN EXPLICIT PROOF OF THE GENERALIZED GAUSS-BONNET FORMULA

by

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To Jean-Michel Bismut on the occasion of his 60th birthday

Abstract. — In this paper we construct an explicit representative for the Grothendieck fundamental class $[Z] \in \mathrm{Ext}^r(\mathcal{O}_Z, \Omega_X^r)$ of a complex submanifold Z of a complex manifold X when Z is the zero locus of a real analytic section of a holomorphic vector bundle E of rank r on X. To this data we associate a super-connection A on $\bigwedge^* E^\vee$, which gives a "twisted resolution" T^* of $\mathcal{O}_Z$ such that the "generalized super-trace" of $\frac{1}{r!}A^{2r}$, which is a map of complexes from T^* to the Dolbeault complex $\mathcal{A}_X^{r,*}$, represents $[Z]$. One may then read off the Gauss-Bonnet formula from this map of complexes.

Résumé **(Une démonstration explicite de la formule de Gauss-Bonnet généralisée)**
Dans cet article nous construisons un représentant explicite de la classe fondamentale de Grothendieck $[Z] \in \mathrm{Ext}^r(\mathcal{O}_Z, \Omega_X^r)$ d'une sous-variété Z dans une variété lisse complexe X quand Z est le lieu des zéros d'une section réelle analytique d'un fibré vectoriel holomorphe E de rang r sur X. Nous associons à cette donnée une super-connection A sur $\bigwedge^* E^\vee$, qui fournit une « résolution tordue » T^* de $\mathcal{O}_Z$ telle que la « super-trace généralisée » de $\frac{1}{r!}A^{2r}$, qui est un morphisme de complexes de T^* vers le complexe de Dolbeault $\mathcal{A}_X^{r,*}$, représente $[Z]$. On peut alors lire la formule de Gauss-Bonnet à partir de cette application entre complexes.

Introduction

If X is a complex manifold, and τ is a holomorphic section, transverse to the zero section, of the dual $E^\vee$ of a rank r holomorphic vector bundle, it is well known that

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the fundamental class of the locus Z of zeros of τ is equal to the top Chern class of the bundle $E^\vee$:

$$[Z] = c_r(E^\vee) = (-1)^r c_r(E)$$

For Hodge cohomology, this is the fact that the image of the Grothendieck fundamental class

$$[Z] \in \mathrm{Ext}^r(\mathcal{O}_Z, \Omega_X^r)$$

under the map

$$\mathrm{Ext}^r(\mathcal{O}_Z, \Omega_X^r) \to \mathrm{Ext}^r(\mathcal{O}_X, \Omega_X^r) = \mathrm{H}^r(X, \Omega_X^r)$$

coincides with the top Chern class of $E^\vee$. Proofs of this result tend to be indirect, i.e. they depend on the axioms for cycle classes and Chern classes, and comparison with "standard" cases.

However, one may observe that the section τ gives rise to an explicit global Koszul resolution

$$K^*(\tau) = (\textstyle\bigwedge^* E^\vee, \iota_\tau) \to \mathcal{O}_Z\ ,$$

and so the theorem can be rephrased as saying that image of $[Z]$ under the map:

$$\mathrm{Ext}^r(K^*(\tau), \Omega_X^r) \to \mathrm{Ext}^r(\mathcal{O}_X, \Omega_X^r)$$

induced by the isomorphism $\mathcal{O}_X \simeq K^0(\tau)$, is the top Chern class of $E^\vee$. Our first result is to show that a choice of connection $\widetilde{\nabla}$ on E, determines, via Chern-Weil theory applied to superconnections, an *explicit* map of complexes from the Koszul complex $K^*(\tau)$ to the Dolbeault complex of Ω_X^r, which represents the Grothendieck fundamental class and the restriction of which to the degree zero component $\mathcal{O}_X$ of the Koszul complex is *precisely* multiplication by the r-th Chern form of $E^\vee$.

One motivation for the current paper was to obtain a better understanding of the proof by Toledo and Tong of the Hirzebruch-Riemann-Roch theorem in **[12]**. In that paper the authors used local Koszul resolutions of the structure sheaf of the diagonal $\Delta_X \subset X \times X$ to construct the Grothendieck fundamental class $[\Delta_X]$, and then to compute $\chi(X, \mathcal{O}_X)$ as the degree of the restriction of the appropriate Kunneth-component of $[\Delta_X]$ to the diagonal. For such a computation one needs only the existence of a "nice" representative of the Grothendieck fundamental class in some neighborhood of the diagonal. However the diagonal Δ_X is *not* in general the zero set of a holomorphic section of a vector bundle. Instead one can use the "holomorphic exponential map" (see the article **[10]** for an exposition) to construct, in a neighborhood of the diagonal, a *real analytic* section of $p^*(T_X)$, which vanishes exactly on the diagonal. (Here $p : X \times X \to X$ is the projection onto the first factor.) Thus we are led to consider what happens if we ask only that τ be real analytic rather than holomorphic. In our second main result, we use the theory of superconnections and twisted complexes in the style of Brown **[5]**, and of Toledo and Tong (*op. cit.*) to construct a map from the Dolbeault resolution of $K^*(\tau)$ to that of Ω_X^r representing the Grothendieck fundamental class and which restricts to the r-th Chern form of $E^\vee$. An important tool in this construction is a non-commutative version of the supertrace for endomorphisms of Grassman algebras.

We should also remark that instead of working in the real analytic category, one can make a very similar argument in the algebraic category, using formal schemes.

Let us now give a more detailed outline of the paper. Recall that the section τ gives rise to a natural Koszul resolution $K(\tau)^* \to \mathcal{O}_Z$, in which $K(\tau)^{-j} = \bigwedge^j \mathcal{E}$. Here $\mathcal{E}$ is the sheaf of holomorphic sections of E. Choose a connection $\nabla : \mathcal{A}_X \otimes \mathcal{E} \to \mathcal{A}_X^{1,0} \otimes \mathcal{E}$ of type $(1,0)$ ($\mathcal{A}_X^{1,0}$ being the sheaf of real analytic $(1,0)$-forms on X) on $\mathcal{E}$, such that $\nabla^2 = 0$. Let $\widetilde{\nabla} = \nabla + \overline{\partial}$ be the associated connection. We view ∇ as acting not only on $\mathcal{E}$, but on all tensor constructions on $\mathcal{E}$. Then our first result is:

***Theorem* (A).** — *The connection ∇ and the section τ determine a map of complexes, from the Koszul resolution $K(\tau)^*$ of $\mathcal{O}_Z$, to the Dolbeault resolution $\mathcal{A}_X^{r,*}[r]$ of $\Omega_X^r[r]$*

$$\psi : K(\tau)^* \to \mathcal{A}_X^{r,*}[r]$$

the degree $-r$ component of which is $\frac{1}{r!}(\imath_{\nabla(\tau)})^r$, and the degree 0 component $K(\tau)^0 = \mathcal{O}_X \to \mathcal{A}_X^{r,*}[r]^0 = \mathcal{A}_X^{r,r}$ of which is represented by the r-th Chern form of $(E^\vee, \widetilde{\nabla})$. In general ψ is given by a linear algebra construction involving ∇ and the curvature $R = [\nabla, \overline{\partial}]_s$ of $\widetilde{\nabla}$, and we have:

- *The class in $\operatorname{Ext}^r_{\mathcal{O}_X}(\mathcal{O}_Z, \Omega^r_X)$ represented by ψ is the Grothendieck fundamental class $[Z]$.*
- *The image of $[Z]$ in $\operatorname{Ext}^r_{\mathcal{O}_X}(\mathcal{O}_X, \Omega^r_X) \simeq H^{r,r}(X, \mathbb{C})$, is represented by the degree zero component of ψ, which is equal to the r-th Chern form $c_r(E^\vee, \widetilde{\nabla})$*

It follows immediately that the image of $[Z]$ in $H^{r,r}(X, \mathbb{C})$ is equal to $c_r(E^\vee)$.

The proof of Theorem A is contained in Section 5. (cf. Theorem 5.5 and Corollary 5.6).

In the second half of the paper, we extend Theorem A to the case where Z is the zero locus of a real analytic section of $E^\vee$. It is no longer the case that τ determines a Koszul resolution of $\mathcal{O}_Z$, but instead we get a resolution of $\mathcal{A}_X^{0,*} \otimes \mathcal{O}_Z$. In order to get a complex that is quasi-isomorphic to $\mathcal{O}_Z$, we construct a resolution of the Dolbeault resolution $\mathcal{A}_X^{0,*} \otimes \mathcal{O}_Z$ of $\mathcal{O}_Z$, by constructing a *twisted differential*, δ, in the sense of Toledo and Tong [**13**], on $\mathcal{A}_X^{0,*} \otimes \bigwedge^* \mathcal{E}$.

A key tool in extending Theorem A to this situation is the notion of the "generalized supertrace" of an endomorphism of the exterior algebra of a finitely generated projective module. Suppose that V is a (locally) free module of finite rank r over a commutative ring k. Then the generalized supertrace is a map

$$\operatorname{Tr}_\Lambda : \operatorname{End}_k(\textstyle\bigwedge^* V) \to \bigwedge^* V^\vee$$

(c.f. Definition 6.1). If A is a graded-commutative algebra over k, we can extend this to a map

$$\operatorname{Tr}_\Lambda : \operatorname{End}_A(A \widehat{\otimes} \textstyle\bigwedge^* V) \to A \widehat{\otimes} \bigwedge^* V^\vee$$

Here $\widehat{\otimes}$ denotes the "super" or graded tensor product. If $\varphi \in \operatorname{End}_A(\bigwedge^* V)$, then the degree 0 component of $\operatorname{Tr}_\Lambda(\varphi)$ is the usual super-trace of φ. The key property of $\operatorname{Tr}_\Lambda$ (which is proved in Section 3.) is:

Proposition. — *Assume that* $\varphi \in \mathrm{End}_A(\bigwedge^* V)$*, and let* $\delta \in \mathrm{End}_A(\bigwedge^* V)$ *be an* A*-linear superderivation. Then*

$$\mathrm{Tr}_\Lambda[\delta, \varphi]_s = [\delta, \mathrm{Tr}_\Lambda(\varphi)]_s$$

***Theorem* (B)**. — *Let* Z *be a complex submanifold of* X *such that there exists a holomorphic vector bundle* $\pi : E \to X$ *and* $\tau \in \Gamma(X, \mathcal{A}_X \otimes \mathcal{E}^\vee)$ *such that* $\imath_\tau : \mathcal{A}_X \otimes \mathcal{E} \to \mathcal{A}_X \otimes \mathcal{I}_Z$ *is surjective. Then*

- *There is a superconnection* δ *of type (0,1), on the super-bundle* $\bigwedge^* E$*, such that:*
  1. $\delta^2 = 0$*, so* δ *defines a differential on* $\mathcal{A}_X^{0,*} \otimes \bigwedge^* \mathcal{E}$*,*
  2. *the component of* δ *of degree* -1 *with respect to the grading on* $\bigwedge^* E$ *is the Koszul differential* $\imath_\tau$*,*
  3. *If we write* δ *for the induced differential on* $\mathcal{A}_X^{0,*} \otimes \bigwedge^* \mathcal{E}$*, then the map* $\bigwedge^0 \mathcal{E} = \mathcal{O}_X \to \mathcal{O}_Z$ *induces a quasi-isomorphism of complexes:*

$$(\mathcal{A}_X^{0,*} \otimes \textstyle\bigwedge^* \mathcal{E}, \delta) \xrightarrow{\sim} (\mathcal{A}_X^{0,*} \otimes \mathcal{O}_Z, \overline{\partial}) \xleftarrow{\sim} \mathcal{O}_Z$$

- *Let* R_A *be the curvature of the superconnection* $A = \nabla + \delta$ *on* $\bigwedge^* E$*. Then the generalized supertrace of* $\frac{1}{r!}R_A^r$ *defines a map of complexes*

$$\mathcal{A}_X^{0,*} \otimes \textstyle\bigwedge^* \mathcal{E} \to \mathcal{A}_X^{r,*}[r],$$

  which, via the quasi-isomorphisms in part 1), represents the Grothendieck fundamental class $[Z]$*,*
- *The image of* $[Z]$ *in* $H^{r,r}(X, \mathbb{C})$ *is represented by the degree 0 component of the generalized supertrace of* $\frac{1}{r!}R_A^r$*, i.e., by the super-trace of* $\frac{1}{r!}R_A^r$*, which by Quillen* **[11]** *is an* (r,r)*-form representing the Chern character* $\mathrm{ch}_r(\bigwedge^* E)$*.*

The proof of Theorem B is contained in Proposition 8.4, Theorem 10.3, and Corollary 10.5. We would like to thank the referee for comments which let to a substantial improvement in the organization of the paper.

1. Superobjects

Throughout this paper we will use the language of *super-objects*. We include here basic definitions and properties for the convenience of the reader and to fix notation. We omit the details and proofs, which may be found in **[11]** and **[4]**.

Let k be a commutative ring with unity .

Definition 1.1. — *A* k*-module* V *with a* $\mathbb{Z}/2\mathbb{Z}$*-grading is called a* k*-supermodule.*

Remark 1.2. — In the same spirit, a $\mathbb{Z}/2\mathbb{Z}$-graded object in an additive category is called a *superobject*. As realizations of this general definition, we will be dealing with super algebras, super vector bundles on a smooth manifold, and sheaves of superalgebras on a topological space, etc.

We will write V^+ and V^- for the degree 0 (mod 2) and degree 1 (mod 2) parts of V and we will call them the even and the odd parts of V respectively. Let $\nu \in V$ be a homogeneous element. We say $|\nu| = 0$ if $\nu \in V^+$ and $|\nu| = 1$ if $\nu \in V^-$.

$\mathrm{End}_k(V)$ is also a k-supermodule with the grading

$$\begin{aligned} \mathrm{End}_k(V)^+ &= \mathrm{Hom}_k(V^+, V^+) \oplus \mathrm{Hom}_k(V^-, V^-) \\ \mathrm{End}_k(V)^- &= \mathrm{Hom}_k(V^+, V^-) \oplus \mathrm{Hom}_k(V^-, V^+) \end{aligned}$$

Moreover, the algebra of endomorphisms $\mathrm{End}_k(V)$ is a k-superalgebra with this grading. If no confusion is likely to arise, we will suppress the mention of the ring k from now on.

Definition 1.3. — *Let A be a superalgebra. The supercommutator of two elements of A is*

$$[a, b]_s = ab - (-1)^{|a||b|} ba$$

where a and b are homogeneous. The supercommutator is extended bilinearly to non-homogeneous a and b.

If the supercommutator $[\,,]_s : A \otimes A \to A$ is the zero map, then A is called a commutative superalgebra. The exterior algebra of a free module M with the $\mathbb{Z}/2\mathbb{Z}$-grading $\bigwedge^+ M = \bigoplus_{p\ even} \bigwedge^p M$ and $\bigwedge^- M = \bigoplus_{p\ odd} \bigwedge^p M$ is a commutative superalgebra.

Let V be finitely generated and projective. Assume that $\frac{1}{2} \in k$. Giving a $\mathbb{Z}/2\mathbb{Z}$-grading on V is equivalent to giving an involution $\epsilon \in \mathrm{End}_k(V)$, that is $\epsilon^2 = I$. The even and the odd parts are the eigenspaces corresponding to the eigenvalues $+1$ and -1 respectively. In the same fashion, the $\mathbb{Z}/2\mathbb{Z}$-grading on $\mathrm{End}_k(V)$ can be given by the involution

$$\rho(\varphi) = \epsilon \circ \varphi \circ \epsilon$$

where $\varphi \in \mathrm{End}_k(V)$.

Definition 1.4. — *Let $\varphi \in \mathrm{End}_k(V)$. The supertrace of φ, denoted by $\mathrm{tr}_s(\varphi)$, is defined to be*

$$\mathrm{tr}_s(\varphi) = \mathrm{tr}(\epsilon \circ \varphi)$$

where 'tr' is the usual trace map.

Lemma 1.5. — *The supertrace vanishes on supercommutators.*

Proof. — Cf. [**11**]. □

Let A and B be superalgebras. We define the *super tensor product* of A and B, denoted by $A \widehat{\otimes} B$, to be the k-module $A \otimes B$ with the $\mathbb{Z}/2\mathbb{Z}$-grading

$$\begin{aligned} (A \widehat{\otimes} B)^+ &= (A^+ \otimes B^+) \oplus (A^- \otimes B^-) \\ (A \widehat{\otimes} B)^- &= (A^+ \otimes B^-) \oplus (A^- \otimes B^+) \end{aligned}$$

and the algebra structure

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = (-1)^{|b_1||a_2|} a_1 a_2 \otimes b_1 b_2$$