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TORSION INVARIANTS FOR FAMILIES

by

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Dedicated to Jean-Michel Bismut on the occasion of his 60^th birthday

Abstract. — We give an overview over the higher torsion invariants of Bismut-Lott, Igusa-Klein and Dwyer-Weiss-Williams, including some more or less recent developments.

***Résumé* (Invariants de torsion en familles).** — On expose la théorie des invariants de torsion supérieures de Bismut-Lott, Igusa-Klein et Dwyer-Weiss-Williams, ainsi que ses développements récents.

The classical Franz-Reidemeister torsion τ_{FR} is an invariant of manifolds with acyclic unitarily flat vector bundles [**62**], [**33**]. In contrast to most other algebraic-topological invariants known at that time, it is invariant under homeomorphisms and simple-homotopy equivalences, but not under general homotopy equivalences. In particular, it can distinguish homeomorphism types of homotopy-equivalent lens spaces. Hatcher and Wagoner suggested in [**39**] to extend τ_{FR} to families of manifolds $p\colon E \to B$ using pseudoisotopies and Morse theory. A construction of such a higher Franz-Reidemeister torsion τ was first proposed by John Klein in [**48**] using a variation of Waldhausen's A-theory. Other descriptions of τ were later given by Igusa and Klein in [**45**], [**46**].

In this overview, we will refer to the construction in [**42**]. Let $p\colon E \to B$ be a family of smooth manifolds, and let $F \to E$ be a unitarily flat complex vector bundle of rank r such that the fibrewise cohomology with coefficients in F forms a unipotent bundle over B. Using a function $h\colon E \to \mathbb{R}$ that has only Morse and

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birth-death singularities along each fibre of p, and with trivialised fibrewise unstable tangent bundle, one constructs a homotopy class of maps $\xi_h(M/B;F)$ from B to a classifying space $Wh^h(M_r(\mathbb{C}),U(r))$. Now, the higher torsion $\tau(E/B;F) \in H^{4\bullet}(B;\mathbb{R})$ is defined as the pull-back of a certain universal cohomology class $\tau \in H^{4\bullet}\big(Wh^h(M_r(\mathbb{C}),U(r));\mathbb{R}\big)$.

On the other hand, Ray and Singer defined an analytic torsion $\mathcal{T}_{\mathrm{RS}}$ of unitarily flat complex vector bundles on compact manifolds in [**61**] and conjectured that $\mathcal{T}_{\mathrm{RS}} = \tau_{\mathrm{FR}}$. This conjecture was established independently by Cheeger [**26**] and Müller [**59**]. The most general comparison result was given by Bismut and Zhang in [**17**] and [**18**]. In [**64**], Wagoner predicted the existence of a "higher analytic torsion" that detects homotopy classes in the diffeomorphism groups of smooth closed manifolds. Such an invariant was defined later by Bismut and Lott in [**16**].

Kamber and Tondeur constructed characteristic classes $\mathrm{ch}^{\mathrm{o}}(F) \in H^{\mathrm{odd}}(M;\mathbb{R})$ of flat vector bundles $F \to M$ in [**47**] that provide obstructions towards finding a parallel metric. If $p\colon E \to B$ is a smooth bundle of compact manifolds and $F \to E$ is flat, Bismut and Lott proved a Grothendieck-Riemann-Roch theorem relating the characteristic classes of F to those of the fibrewise cohomology $H(E/B;F) \to B$. The higher analytic torsion form $\mathcal{T}(T^HE, g^{TX}, g^F)$ appears in a refinement of this theorem to the level of differential forms. Its component in degree 0 equals the Ray-Singer analytic torsion of the fibres, and the refined Grothendieck-Riemann-Roch theorem implies a variation formula for the Ray-Singer torsion that was already discovered in [**17**].

In [**30**], Dwyer, Weiss and Williams gave yet another approach to higher torsion. They defined three generalised Euler characteristics for bundles $p\colon E \to B$ of homotopy finitely dominated spaces, topological manifolds, and smooth manifolds, respectively, with values in certain bundles over B. A flat complex vector bundle $F \to E$ defines a homotopy class of maps from E to the algebraic K-theory space $K(\mathbb{C})$. The Euler characteristics above give analogous maps $B \to K(\mathbb{C})$ for the fibrewise cohomology $H(E/B;F) \to B$. If F is fibrewise acyclic, these maps lift to three different generalisations of Reidemeister torsion, given again as sections in certain bundles over B. By comparing the three characteristics for smooth manifold bundles, Dwyer, Weiss and Williams also showed that the Grothendieck-Riemann-Roch theorem in [**16**] holds already on the level of classifying maps to $K(\mathbb{C})$.

Bismut-Lott torsion $\mathcal{T}(E/B;F)$ and Igusa-Klein torsion $\tau(E/B;F)$ are very closely related. For particularly nice bundles, this was proved by Bismut and the author in [**12**] and [**36**], [**37**]. We will establish the general case in [**38**]. Igusa also gave a set of axioms in [**44**] that characterise $\tau(E/B;F)$ and hopefully also $\mathcal{T}(E/B;F)$ when F is trivial. Badzioch, Dorabiała and Williams recently gave a cohomological version of the smooth Dwyer-Weiss-Williams torsion in [**3**]. Together with Klein, they proved in [**2**] that it satisfies Igusa's axioms as well. On the other hand, the other two torsions in **[30]** are definitely coarser than Bismut-Lott and Igusa-Klein torsion, because they do not depend on the differentiable structure. They might however be related to the Bismut-Lott or Igusa-Klein torsion of a virtual flat vector bundle F of rank zero, see Remark 7.5 below.

Let us now recall one of the most import applications of higher torsion invariants. It is possible to construct two smooth manifold bundles $p_i\colon E_i \to B$ for $i = 0$, 1 with diffeomorphic fibres, such that there exists a homeomorphism $\varphi\colon E_0 \to E_1$ with $p_0 = p_1 \circ \varphi$ and with a lift to an isomorphism of vertical tangent bundles, but no such diffeomorphism. The first example of such bundles p_i was constructed by Hatcher, and it was later proved by Bökstedt that p_0 and p_1 are not diffeomorphic in the sense above **[19]**. Igusa showed in **[42]** that the higher torsion invariants $\tau(E_i/B;\mathbb{C})$ differ, and by **[36]**, the Bismut-Lott torsions $\mathcal{T}(E_i/B;\mathbb{C})$ differ as well. Hatcher's example can be generalised to construct many different smooth structures on bundles $p\colon E \to B$. We expect that higher torsion invariants distinguish many of these different structures, but not all of them.

One may wonder why one wants to consider so many different higher torsion invariants, in particular, if some of them are conjectured to provide the same information. We will see that different constructions of these invariants give rise to different applications. Since Hatcher's example and its generalisations come with natural fibrewise Morse functions, the difference of the Igusa-Klein torsions of different smooth structures is sometimes easy to compute. Due to Igusa's axiomatic approach, one can also understand the topological meaning of Igusa-Klein torsion. On the other hand, one can classify smooth structures on a topological manifold bundle $p\colon E \to B$ in a more abstract way as classes of sections in a certain bundle of classifying spaces over B. These section spaces fit well into the framework of generalised Euler characteristics and Dwyer-Weiss-Williams torsion. But some extra work is necessary to recover cohomological information from this approach.

Finally, Bismut-Lott torsion is defined using the language of local index theory. The proofs of some interesting properties of Bismut-Lott torsion were inspired by parallel results in the setting of the classical Atiyah-Singer family index theorem or the Grothendieck-Riemann-Roch theorem in Arakelov geometry. Bismut-Lott torsion is defined for any flat vector bundle $F \to E$, whereas Igusa-Klein torsion and Dwyer-Weiss-Williams torsion can only be defined if the fibrewise cohomology is of a special type. This makes Bismut-Lott torsion useful for other applications, for example in the definition of a secondary K-theory by Lott **[52]**. Heitsch and Lazarov generalised Bismut-Lott torsion to foliations **[40]**, so one may try to use it to detect different smooth structures on a given foliation, which induce the same structures on the space of leaves. Finally, Bismut and Lebeau recently defined higher torsion invariants using a hypoelliptic Laplacian on the cotangent bundle [**8**], [**15**]. Conjecturally, this torsion can give some information about the fibrewise geodesic flow.

This overview is organised as follows. We start by discussing the index theorem for flat vector bundles by Bismut and Lott in Section 1. In Sections 2 and 3, we introduce Bismut-Lott torsion and state some properties and applications that are inspired by local index theory. In Section 4 and 5, we introduce Igusa-Klein torsion and relate it to Bismut-Lott torsion using two different approaches. Section 6 is devoted to generalised Euler characteristics and Dwyer-Weiss-Williams torsion. In Section 7, we discuss smooth structures on fibre bundles and a possible generalisation to foliations. Finally, we sketch the hypoelliptic operator on the cotangent bundle and its torsion due to Bismut and Lebeau in Section 8.

We have tried to keep the notation and the normalisation of the invariants consistent throughout this paper; as a result, both will disagree with most of the references. In particular, we use the Chern normalisation of [**12**], which is the only normalisation for which Theorem 3.7 and a few other results hold. To keep this paper reasonably short, only the most basic versions of some of the theorems on higher torsion will be explained. Thus we will not discuss some non-trivial generalisations of the theorems below to fibre bundles with group actions. We will also only give hints towards the relation with the classical Atiyah-Singer family index theorem or the Grothendieck-Riemann-Roch theorem in algebraic geometry. Finally, we will not discuss the interesting refinements and generalisations of classical Franz-Reidemeister torsion and Ray-Singer torsion for single manifolds that have been invented in the last few years.

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1. An Index Theorem for Flat Vector Bundles

There exists a theory of characteristic classes of flat vector bundles that is parallel to the theory of Chern classes and Chern-Weil differential forms. These classes have been constructed by Kamber and Tondeur [**47**], and are closely related to the classes used by Borel [**20**] to study the algebraic K-theory of number fields.

Analytic torsion forms made their first appearance in a local index theorem for these Kamber-Tondeur classes by Bismut and Lott [**16**]. Refinements of this theorem have later been given by Dwyer, Weiss and Williams [**30**] and by Bismut [**7**] and Ma and Zhang [**56**].

1.1. Characteristic classes for flat vector bundles. — Before we introduce Kamber-Tondeur forms, let us first recall classical Chern-Weil theory. Let $V \to M$ be a complex vector bundle, and let ∇^V be a connection on V with curvature $(\nabla^V)^2 \in \Omega^2(M; \operatorname{End} V)$. Then one defines the Chern character form

$$\mathrm{ch}\big(V, \nabla^V\big) = \mathrm{tr}_V\Big(e^{-\frac{(\nabla^V)^2}{2\pi i}}\Big) \in \Omega^{\mathrm{even}}(M; \mathbb{C})\ . \tag{1.1}$$

This form is closed because the covariant derivative $[\nabla^V, (\nabla^V)^2]$ of the curvature vanishes by the Bianchi identity, so

$$d\,\mathrm{ch}\big(V, \nabla^V\big) = \mathrm{tr}_V\Big(\Big[\nabla^V, e^{-\frac{(\nabla^V)^2}{2\pi i}}\Big]\Big) = 0\ . \tag{1.2}$$

If $\nabla^{V,0}$ and $\nabla^{V,1}$ are two connections on V, one can choose a connection $\nabla^{\tilde V}$ on the natural extension $\tilde V$ of V to $M \times [0,1]$ with $\nabla^{\tilde V}|_{M\times\{i\}} = \nabla^{V,i}$ for $i = 0$, 1. Stokes' theorem then implies

$$\begin{aligned} \mathrm{ch}\big(V, \nabla^{V,1}\big) - \mathrm{ch}\big(V, \nabla^{V,0}\big) &= d\,\widetilde{\mathrm{ch}}\big(V, \nabla^{V,0}, \nabla^{V,1}\big), \\ \text{with} \qquad \widetilde{\mathrm{ch}}\big(V, \nabla^{V,0}, \nabla^{V,1}\big) &= \int_0^1 \iota_{\frac{\partial}{\partial t}}\,\mathrm{ch}\Big(\tilde V, \nabla^{\tilde V}\Big)\,dt. \end{aligned} \tag{1.3}$$

Thus, the class $\mathrm{ch}(V)$ of $\mathrm{ch}(V, \nabla^V)$ in de Rham cohomology is independent of ∇^V. Moreover, $\widetilde{\mathrm{ch}}(V, \nabla^{V,0}, \nabla^{V,1})$ is independent of the choice of $\nabla^{\tilde V}$ up to an exact form.

Now let $F \to M$ be a flat vector bundle, so F comes with a fixed connection ∇^F such that $(\nabla^F)^2 = 0$. We choose a metric g^F on F and define the adjoint connection $\nabla^{F,*}$ with respect to g^F such that

$$dg(v, w) = g\big(\nabla^F v, w\big) + g\big(v, \nabla^{F,*} w\big) \tag{1.4}$$

for all sections v, w of F. Then the form

$$\mathrm{ch}^\circ\big(F, g^F\big) = \pi i\,\widetilde{\mathrm{ch}}\big(F, \nabla^F, \nabla^{F,*}\big) \in \Omega^{\mathrm{odd}}(M; \mathbb{R}) \tag{1.5}$$

is real, odd and also closed, because

$$d\,\mathrm{ch}^\circ\big(F, g^F\big) = \pi i\,\mathrm{ch}\big(F, \nabla^{F,*}\big) - \pi i\,\mathrm{ch}\big(F, \nabla^F\big) = 0\ . \tag{1.6}$$

Clearly, if g^F is parallel with respect to ∇^F, then $\mathrm{ch}^\circ(F, g^F) = 0$.

Let $g^{F,0}$, $g^{F,1}$ be two metrics on F. Proceeding as in (1.3), one constructs a form $\widetilde{\mathrm{ch}}^\circ(F, g^{F,0}, g^{F,1}) \in \Omega^{\mathrm{even}}(M)$ such that

$$\mathrm{ch}^\circ\big(F, g^{F,1}\big) - \mathrm{ch}^\circ\big(F, g^{F,0}\big) = d\,\widetilde{\mathrm{ch}}^\circ\big(F, g^{F,0}, g^{F,1}\big)\ . \tag{1.7}$$