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INDEX OF TRANSVERSALLY ELLIPTIC OPERATORS

by

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Dedicated to Jean-Michel Bismut on the occasion of his 60th birthday

Abstract. — In the 70's, the notion of analytic index has been extended by Atiyah and Singer to the class of transversally elliptic operators. They did not, however, give a general cohomological formula for the index. This was accomplished many years later by Berline and Vergne. The Berline-Vergne formula is an integral of a non-compactly supported equivariant form on the cotangent bundle, and depends on rather subtle growth conditions for these forms.

This paper gives an alternative expression for the index, where the non-compactly supported form is replaced with a compactly supported one, but with generalized coefficients.

***Résumé* (Indice d'opérateurs transversalement elliptiques).** — Dans les années 70, Atiyah et Singer ont étendu la notion d'indice analytique au cadre des opérateurs transversalement elliptiques. Néanmoins, ils ne donnaient pas de formule cohomologique générale pour cet indice. Ce problème a été résolu bien des années après par Berline et Vergne. La formule de Berline-Vergne exprime l'indice comme l'intégrale sur un fibré cotangent d'une forme équivariante à support non-compact: ici une propriété de croissance très particulière de cette forme est requise pour assurer l'existence de l'intégrale.

Le but de ce travail est de donner une autre formulation de cet indice, où la forme équivariante à support non-compact est remplacée par une forme équivariante à support compact, mais avec des coefficients généralisés.

1. Introduction

Let M be a compact manifold. The Atiyah-Singer formula for the index of an *elliptic* pseudo-differential operator P on M with *elliptic* symbol σ on $\mathbf{T}^*M$ involves

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integration over the non compact manifold $\mathbf{T}^*M$ of the Chern character $\mathrm{Ch}_c(\sigma)$ of σ multiplied by the square of the $\widehat{\mathrm{A}}$-genus of M:

$$\mathrm{index}(P) = (2i\pi)^{-\dim M} \int_{\mathbf{T}^*M} \widehat{\mathrm{A}}(M)^2 \,\mathrm{Ch}_c(\sigma).$$

Here σ, the principal symbol of P, is a morphism of vector bundles on $\mathbf{T}^*M$ invertible outside the zero section of $\mathbf{T}^*M$ and the Chern character $\mathrm{Ch}_c(\sigma)$ is supported on a compact neighborhood of M embedded in $\mathbf{T}^*M$ as the zero section. It is important that the representative of the Chern character $\mathrm{Ch}_c(\sigma)$ is compactly supported to perform integration.

Assume that a compact Lie group K (with Lie algebra $\mathfrak{k}$) acts on M. If the elliptic operator P is K-invariant, then $\mathrm{index}(P)$ is a smooth function on K. The equivariant index of P can be expressed similarly as the integral of the equivariant Chern character of σ multiplied by the square of the equivariant $\widehat{\mathrm{A}}$-genus of M: for $X \in \mathfrak{k}$ small enough,

$$\mathrm{index}(P)(\mathrm{e}^X) = (2i\pi)^{-\dim M} \int_{\mathbf{T}^*M} \widehat{\mathrm{A}}(M)^2(X) \,\mathrm{Ch}_c(\sigma)(X). \tag{1}$$

Here $\mathrm{Ch}_c(\sigma)(X)$ is a compactly supported closed equivariant differential form, that is a differential form on $\mathbf{T}^*M$ depending smoothly of $X \in \mathfrak{k}$, and closed for the equivariant differential D. The result of the integration determines a smooth function on a neighborhood of e in K and similar formulae can be given near any point of K. Formula (1) is a "delocalization" of the Atiyah-Bott-Segal-Singer formula, in the sense of Bismut [**9**].

The delocalized index formula (1) can be adapted to new cases such as:

- Index of transversally elliptic operators.
- L^2-index of some elliptic operators on some non-compact manifolds (Rossmann formula for discrete series [**20**]).

Indeed, in these two contexts, the index exists in the sense of generalized functions but cannot be always computed in terms of fixed point formulae. A "delocalized" formula will however continue to have a meaning, as we explain now for transversally elliptic operators.

The invariant operator P with symbol $\sigma(x,\xi)$ on $\mathbf{T}^*M$ is called transversally elliptic, if it is elliptic in the directions transverse to K-orbits. In this case, the operator P has again an index which is a generalized function on K [**1**]. A very simple example of transversally elliptic operator is the operator 0 on $L^2(K)$: its index is the trace of the action of K in $L^2(K)$, that is the δ-function on K. At the opposite side, K-invariant elliptic operators are of course transversally elliptic, and index of such operators are smooth functions on K given by Formula (1). Thus a cohomological formula must incorporate these two extreme cases. Such a cohomological formula was given in Berline-Vergne [**7, 8**]. We present here a new point of view, where the equivariant Chern character $\mathrm{Ch}_c(\sigma)(X)$ entering in Formula (1) is replaced by a Chern character with generalized coefficients, but still *compactly supported.* Let us briefly explain the construction.

Let $\mathbf{T}^*_K M \subset \mathbf{T}^* M$ be the cone formed by the covectors $\xi \in \mathbf{T}^*_x M$ which vanish on tangent vectors to the orbit $K \cdot x$. Let $\mathrm{supp}(\sigma)$ be the support of the symbol σ of a transversally elliptic operator P. By definition, the intersection $\mathrm{supp}(\sigma) \cap \mathbf{T}^*_K(M)$ is compact. By the Quillen super-connection construction, the Chern character $\mathrm{Ch}(\sigma)(X)$ is a closed equivariant differential form supported near the closed set $\mathrm{supp}(\sigma)$. Using the Liouville 1-form ω of $\mathbf{T}^* M$, we construct a closed equivariant form $\mathrm{One}(\omega)$ supported near $\mathbf{T}^*_K M$. Outside $\mathbf{T}^*_K M$, one has indeed the equation $1 = D(\frac{\omega}{D\omega})$, where the inverse of the form $D\omega$ is defined by $-i\int_0^\infty e^{itD\omega}dt$, an integral which is well defined in the generalized sense, that is tested against a smooth compactly supported density on $\mathfrak{k}$. Thus using a function χ equal to 1 on a small neighborhood of $\mathbf{T}^*_K M$, the closed equivariant form

$$\mathrm{One}(\omega)(X) = \chi + d\chi \frac{\omega}{D\omega(X)}, \quad X \in \mathfrak{k},$$

is well defined, supported near $\mathbf{T}^*_K M$, and represents 1 in cohomology. Remark that

$$\mathrm{Ch}_c(\sigma, \omega) := \mathrm{Ch}(\sigma)(X)\mathrm{One}(\omega)(X)$$

is *compactly supported.* We prove that, for $X \in \mathfrak{k}$ small enough, we have

$$\text{(2)} \qquad \mathrm{index}(P)(\mathrm{e}^X) = (2i\pi)^{-\dim M} \int_{\mathbf{T}^* M} \widehat{\mathrm{A}}(M)^2(X)\, \mathrm{Ch}(\sigma)(X)\, \mathrm{One}(\omega)(X).$$

This formula is thus similar to the delocalized version of the Atiyah-Bott-Segal-Singer equivariant index theorem. We have just localized the formula for the index near $\mathbf{T}^*_K M$ with the help of the form $\mathrm{One}(\omega)$, equal to 1 in cohomology, but supported near $\mathbf{T}^*_K M$.

When P is elliptic we can furthermore localize on the zeros of VX (the vector field on M produced by the action of X) and we obtain the Atiyah-Bott-Segal-Singer fixed point formulae for the equivariant index of P. However the main difference is that usually we cannot obtain a fixed point formula for the index. For example, the index of a transversally elliptic operator P where K acts freely is a generalized function on K supported at the origin. Thus in this case the use of the form $\mathrm{One}(\omega)$ is essential. Its role is clearly explained in the example of the 0 operator on S^1 given at the end of this introduction.

We need also to define the formula for the index at any point $s \in K$, in terms of integrals over $\mathbf{T}^* M(s)$, where $M(s)$ is the fixed point submanifold of M under the action of s. The compatibility properties (descent method) between the formulae at different points s are easy to prove, thanks to a localization formula adapted to this generalized setting.

In the Berline-Vergne cohomological formula for the index of P, the Chern character $\mathrm{Ch}_c(\sigma)(X)$ in Formula (1) was replaced by a Chern character $\mathrm{Ch}_{BV}(\sigma,\omega)(X)$ depending also of the Liouville 1-form ω. This Chern character $\mathrm{Ch}_{BV}(\sigma,\omega)$ is constructed for "good symbols" σ. It looks like a Gaussian in the transverse directions, and is oscillatory in the directions of the orbits. Our new point of view defines the *compactly supported* product class $\mathrm{Ch}(\sigma)\mathrm{One}(\omega)$ in a straightforward way. We proved in [**17**] that the classes $\mathrm{Ch}_{BV}(\sigma,\omega)$ and $\mathrm{Ch}(\sigma)\mathrm{One}(\omega)$ are equivalent in an appropriate cohomology space, so that our new cohomological formula gives the analytic index. Nevertheless, we will see in this paper that it is technically simpler to work with the compactly supported equivariant form $\mathrm{Ch}(\sigma)\mathrm{One}(\omega)$ rather than with the equivariant form $\mathrm{Ch}_{BV}(\sigma,\omega)$ which has subtle growth on the cotangent bundle. So we choose to prove directly the equality between the analytic index and the cohomological index, and we show that our formula in terms of the product class $\mathrm{Ch}(\sigma)\mathrm{One}(\omega)$ is natural. We follow the same line as Atiyah-Singer: functoriality with respect to products and free actions. The compatibility with the free action reduces basically to the case of the zero operator on K, and the calculation is straightforward. The typical calculation is shown below. The multiplicativity property is more delicate, but is based on a general principle on multiplicativity of relative Chern characters that we proved in a preceding article [**17**]. Thus, following Atiyah-Singer [**1**], we are reduced to the case of S^1 acting on a vector space. The basic examples are then the pushed symbol with index $-\sum_{n=1}^{\infty} e^{in\theta}$ and the index of the tangential $\overline{\partial}$ operators on odd dimensional spheres. We include at the end a general formula due to Fitzpatrick [**11**] for contact manifolds.

Let us finally point out that there are many examples of transversally elliptic operators of great interest. The index of elliptic operators on orbifolds are best understood as indices of transversally elliptic operators on manifolds where a group K acts with finite stabilizers. The restriction to the maximal compact subgroup K of a representation of the discrete series of a real reductive group are indices of transversally elliptic operators [**16**]. More generally, there is a canonical transversally elliptic operator on any prequantized Hamiltonian manifold with proper moment map (under some mild assumptions) [**16**], [**22**]. Furthermore, as already noticed in Atiyah-Singer, and systematically used in [**15**], transversally elliptic operators associated to symplectic vector spaces with proper moment maps and to cotangent manifolds $\mathbf{T}^*K$ are the local building pieces of any K-invariant elliptic operator.

Example 1.1. — *Let us check the validity of* (2) *in the example of the zero operator* 0_{S^1} *from* $S^1\times\mathbb{C}$ *to* $S^1\times\{0\}$*. This operator is* S^1*-transversally elliptic and its index* *is equal to*

$$\delta_1(\mathrm{e}^{iX}) = \sum_{k\in\mathbb{Z}} \mathrm{e}^{ikX}, \quad X \in \mathrm{Lie}(S^1) \simeq \mathbb{R}.$$

The principal symbol σ *of* 0_{S^1} *is the zero morphism* $\mathbf{T}^*S^1 \times \mathbb{C} \to \mathbf{T}^*S^1 \times \{0\}$. *Hence* $\mathrm{Ch}(\sigma)(X) = 1$. *The equivariant class* $\widehat{\mathrm{A}}(S^1)^2(X)$ *is also equal to* 1. *Thus the right hand side of* (2) *becomes*

$$(2i\pi)^{-1} \int_{\mathbf{T}^*S^1} \mathrm{One}(\omega)(X).$$

The cotangent bundle $\mathbf{T}^*S^1$ *is parametrized by* $(\mathrm{e}^{i\theta}, \xi) \in S^1 \times \mathbb{R}$. *The Liouville* 1*-form is* $\omega = -\xi d\theta$ *: the symplectic form* $d\omega = d\theta \wedge d\xi$ *gives the orientation of* $\mathbf{T}^*S^1$. *Since* $VX = -X\frac{\partial}{\partial\theta}$, *we have* $D\omega(X) = d\theta \wedge d\xi - X\xi$.

Let $g \in \mathscr{C}^\infty(\mathbb{R})$ *with compact support and equal to* 1 *in a neighborhood of* 0. *Then* $\chi = g(\xi^2)$ *is a function on* $\mathbf{T}^*S^1$ *which is supported in a neighborhood of* $\mathbf{T}^*_{S^1}S^1 =$ zero section. *We look now at the equivariant form* $\mathrm{One}(\omega)(X) = \chi + d\chi \wedge (-i\omega)\int_0^\infty \mathrm{e}^{itD\omega(X)}\, dt$. *We have*

$$\begin{aligned} \mathrm{One}(\omega)(\mathrm{e}^{i\theta}, \xi, X) &= g(\xi^2) + g'(\xi^2) 2\xi d\xi \wedge (i\xi d\theta) \int_0^\infty \mathrm{e}^{it(d\theta\wedge d\xi - X\xi)}\, dt \\ &= g(\xi^2) - id\theta \wedge d(g(\xi^2)) \left(\int_0^\infty \mathrm{e}^{-itX\xi}\, \xi dt \right). \end{aligned}$$

If we make the change of variable $t\xi \to t$ *in the integral* $\int_0^\infty \mathrm{e}^{-itX\xi}\, \xi dt$ *we get*

$$\mathrm{One}(\omega)(\mathrm{e}^{i\theta}, \xi, X) = \begin{cases} g(\xi^2) - id\theta \wedge d(g(\xi^2)) \left(\int_0^\infty \mathrm{e}^{-itX}\, dt\right), & \text{if} \quad \xi \geq 0; \\ g(\xi^2) + id\theta \wedge d(g(\xi^2)) \left(\int_{-\infty}^0 \mathrm{e}^{-itX}\, dt\right), & \text{if} \quad \xi \leq 0. \end{cases}$$

Finally, since $-\int_{\xi\geq 0} d(g(\xi^2)) = \int_{\xi \leq 0} d(g(\xi^2)) = 1$, *we have*

$$(2i\pi)^{-1} \int_{\mathbf{T}^*S^1} \mathrm{One}(\omega)(X) = \int_{-\infty}^{\infty} \mathrm{e}^{-itX}\, dt.$$

The generalized function $\delta_0(X) = \int_{-\infty}^\infty \mathrm{e}^{-itX}\, dt$ *satisfies*

$$\int_{\mathrm{Lie}(S^1)} \delta_0(X)\varphi(X) dX = \mathrm{vol}(S^1, dX)\varphi(0)$$

for any function $\varphi \in \mathscr{C}^\infty(\mathrm{Lie}(S^1))$ *with compact support. Here* $\mathrm{vol}(S^1, dX) = \int_0^{2\pi} dX$ *is also the volume of* S^1 *with the Haar measure compatible with* dX.

Finally, we see that (2) *corresponds to the following equality of generalized functions*

$$\delta_1(\mathrm{e}^{iX}) = \delta_0(X),$$

which holds for $X \in \mathrm{Lie}(S^1)$ *small enough.*