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CM STABILITY AND THE GENERALIZED FUTAKI INVARIANT II

by

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Dedicated to Jean-Michel Bismut on the occasion of his 60th birthday

Abstract. — The Mabuchi K-energy map is exhibited as a singular metric on the refined CM polarization of any equivariant family $\mathbf{X} \xrightarrow{p} S$. Consequently we show that the generalized Futaki invariant is the leading term in the asymptotics of the reduced K-energy of the generic fiber of the map p. Properness of the K-energy implies that the generalized Futaki invariant is strictly negative.

***Résumé* (CM-stabilité et invariant de Futaki généralisé II).** — On interpréte la K-énergie de Mabuchi comme une métrique singulière sur la CM-polarisation raffinée d'une famille équivariante $\mathbf{X} \xrightarrow{p} S$. Nous montrons que l'invariant de Futaki généralisé est le terme principal de l'asymptotique de la K-énergie réduite de la fibre générique de l'application p. Si la K-énergie est propre, alors l'invariant de Futaki généralisé est strictement négatif.

1. Introduction

1.1. Statement of results. — Throughout this paper $\mathbf{X}$ and S denote smooth, proper complex projective varieties satisfying the following conditions.

1. $\mathbf{X} \subset S \times \mathbb{P}^N$; $\mathbb{P}^N$ denotes the complex projective space of *lines* in $\mathbb{C}^{N+1}$.
2. $p := p_1 : \mathbf{X} \to S$ is flat of relative dimension n, degree d with Hilbert polynomial P.
3. $L|_{\mathbf{X}_z}$ is very ample and the embedding $\mathbf{X}_z := p_1^{-1}(z) \xhookrightarrow{L} \mathbb{P}^N$ is given by a complete linear system for $z \in S$.

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4. There is an action of $G := SL(N+1,\mathbb{C})$ on the data compatible with the projection and the standard action on $\mathbb{P}^N$.

It is well known that (1) and (3) imply that

$$\mathbb{P}(p_{1*}L) \cong S \times \mathbb{P}^N. \tag{1.1}$$

Which in turn is equivalent to the existence of a line bundle $\mathscr{A}$ on S such that

$$p_{1*}L \cong \underbrace{\bigoplus \mathscr{A}}_{N+1}. \tag{1.2}$$

Below $\mathrm{Chow}(\mathbf{X}/S)$ denotes the Chow form of the family $\mathbf{X}/S$, μ is the coefficient of k^{n-1} in $P(k)$, and $\mathscr{M}_n$ is the coefficient of $\binom{m}{n}$ in the CGKM expansion of $\det(p_{1*}L^{\otimes m})$ for $m >> 0$. A complete discussion of these notions is given in "*CM Stability and the Generalized Futaki Invariant I*". We refer the reader to that paper for the basic definitions and constructions that are used in the present article.

We define an invertible sheaf on S as follows.

***Definition 1* (The Refined CM polarization[1]).** — *We have*

$$\mathbb{L}_1(\mathbf{X}/S) := \{\mathrm{Chow}(\mathbf{X}/S) \otimes \mathscr{A}^{d(n+1)}\}^{n(n+1)+\mu} \otimes \mathscr{M}_n^{-2(n+1)} \tag{1.3}$$

With the family $p_1 : \mathbf{X} \to S$ fixed throughout, we will denote $\mathbb{L}_1(\mathbf{X}/S)$ by $\mathbb{L}_1$ in the remainder of the paper.

Our first result exhibits the Mabuchi energy as a *singular* Hermitian metric on $\mathbb{L}_1$.

Theorem 1. — *Let $||\ ||$ be any smooth Hermitian metric on $\mathbb{L}_1^{-1}$.[2] Then there is a continuous function $\Psi_S : S \setminus \Delta \to (-\infty, c)$ such that for all $z \in S/\Delta$*

$$d(n+1)\nu_{\omega|_{\mathbf{X}_z}}(\varphi_\sigma) = \log\left(e^{(n+1)\Psi_S(\sigma z)}\frac{||\ ||^2(\sigma z)}{||\ ||^2(z)}\right). \tag{1.4}$$

Here c denotes a constant which depends only on the choice of background Kähler metrics on S and $\mathbf{X}$, Δ denotes the discriminant locus of the map p_1, and $\omega|_{\mathbf{X}_z}$ denotes the restriction of the Fubini Study form of $\mathbb{P}^N$ to the fiber $\mathbf{X}_z$.

Remark 1. — *This should be compared with the main result in Section 8 of* **[17]**. *The principal contribution of our present work is the observation that the whole theory in Section 8 of* **[17]** *should be recast from the beginning with the sheaf $\mathbb{L}_1$.*

Let $X \hookrightarrow \mathbb{P}^N$ be an n dimensional projective variety with Hilbert polynomial P. Let $Hilb_m(X)$ denote the mth Hilbert point of X (see **[12]** for further information). If λ is a one parameter subgroup of G then it is known (see **[12]**) that the weight,

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(1) We use this terminology in order to distinguish this sheaf from one introduced by the second author in (**[17]**).
(2) $\mathbb{L}_1^{-1}$ denotes the dual of $\mathbb{L}_1$.

$w_\lambda(m)$, of $Hilb_m(X)$ with respect to λ is a *polynomial* in m of degree at most $n+1$. That is,

$$w_\lambda(m) = a_{n+1}(\lambda)m^{n+1} + a_n(\lambda)m^n + \cdots$$

Then the ratio may be expanded as follows.

$$\frac{w_\lambda(m)}{mP(m)} = F_0(\lambda) + F_1(\lambda)\frac{1}{m} + \cdots + F_l(\lambda)\frac{1}{m^l} + \ldots$$

***Definition 2* (Donaldson ([5])).** — *$F_1(\lambda)$ is the generalized Futaki invariant of X with respect to λ.*

In our previous paper we have shown the following.

***Theorem* (The weight of the Refined CM polarization).** — i) *There is a natural G linearization on the line bundle $\mathbb{L}_1$.*

ii) *Let λ be a one parameter subgroup of G. Let $z \in \mathfrak{Hilb}^P_{\mathbb{P}^N}(\mathbb{C})$. Let $w_\lambda(z)$ denote the weight of the restricted $\mathbb{C}^*$ action (whose existence is asserted in i)) on $\mathbb{L}_1^{-1}|_{z_0}$ where $z_0 = \lambda(0)z$. Then*

$$w_\lambda(z) = F_1(\lambda). \tag{1.5}$$

The main result of the paper is the following corollary of (1.4) and (1.5).

***Corollary 1* (Algebraic asymptotics of the Mabuchi energy).** — *Let $\varphi_{\lambda(t)}$ be the Bergman potential associated to an algebraic 1psg λ of G, and let $z \in S \setminus \Delta$. Then there is an asymptotic expansion*

$$d(n+1)\nu_{\omega|_{\mathbf{X}_z}}(\varphi_{\lambda(t)}) - \Psi_S(\lambda(t)) = F_1(\lambda)\log(|t|^2) + O(1) \text{ as } |t| \to 0. \tag{1.6}$$

Moreover $\Psi_S(\lambda(t)) = \psi(\lambda)\log(|t|^2) + O(1)$ where $\psi(\lambda) \in \mathbb{Q}_{\geq 0}$. Moreover, $\psi(\lambda) \in \mathbb{Q}_+$ if and only if $\lambda(0)\mathbf{X}_z = \mathbf{X}_{\lambda(0)z}$ (the limit cycle[(3)] of $\mathbf{X}_z$ under λ) has a component of multiplicity greater than one. Here $O(1)$ denotes any quantity which is bounded as $|t| \to 0$.

Moser iteration and a refined Sobolev inequality (see **[11]** and **[7]**) yield the following.

***Corollary 2*.** — *If $\nu_{\omega|_{\mathbf{X}_z}}$ is proper (bounded from below) then the generalized Futaki invariant of $\mathbf{X}_z$ is strictly negative (nonnegative) for all $\lambda \in G$.*

***Remark 2*.** — *We call the left hand side of (1.6) the* reduced *K-Energy along λ. We also point out that while it is certainly the case that $F_1(\lambda)$ may be defined for any subscheme of $\mathbb{P}^N$ it evidently only controls the behavior of the K-Energy when $\lambda(0)\mathbf{X}_z$ is reduced.*

(3) See **[12]** pg. 61.

Remark 3. — *The precise constant $d(n+1)$ in front of ν_ω is not really crucial, since what really matters is the sign* of $F_1(\lambda)+\psi(\lambda)$. *That $\Psi_S(\lambda(t))$ has logarithmic singularities can be deduced from* [**13**].

Remark 4. — *We emphasize that we do not assume the limit cycle is smooth.*

2. Background and Motivation

Let (X,ω) be a compact Kähler manifold (ω not necessarily a Hodge class) and $P(X,\omega) := \{\varphi \in C^\infty(X) : \omega_\varphi := \omega + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\varphi > 0\}$ the space of Kähler potentials. This is the usual description of all Kähler metrics in the same class as ω (up to translations by constants). It is not an overstatement to say that the most basic problem in Kähler geometry is the following

Does there exist $\varphi \in P(X,\omega)$ such that $\mathrm{Scal}(\omega_\varphi) \equiv \mu$? $(*)$

This is a fully nonlinear *fourth order* elliptic partial differential equation for φ. μ is a constant, the average of the scalar curvature, it depends only on $c_1(X)$ and $[\omega]$. When $c_1(X) > 0$ and ω represents the *anticanonical* class a simple application of the Hodge Theory shows that $(*)$ is equivalent to the *Monge-Ampere equation.*

$$\frac{\det(g_{i\bar{j}} + \varphi_{i\bar{j}})}{\det(g_{i\bar{j}})} = e^{F-\kappa\varphi} \quad (\kappa = 1) \qquad (**)$$

where F denotes the Ricci potential. When $\kappa = 0$ this is the celebrated Calabi problem solved by S.T.Yau and when $\kappa < 0$ this was solved by Aubin and Yau independently in the 70's. It is well known that $(*)$ is actually a *variational* problem. There is a natural energy on the space $P(X,\omega)$ whose critical points are those φ such that ω_φ has constant scalar curvature (csc). This energy was introduced by T. Mabuchi ([**10**]) in the 1980's. It is called the *K-Energy map* (denoted by ν_ω) and is given by the following formula

$$\nu_\omega(\varphi) := -\frac{1}{V}\int_0^1 \int_X \dot{\varphi}_t(\mathrm{Scal}(\varphi_t) - \mu)\omega_t^n dt.$$

Above, φ_t is a smooth path in $P(X,\omega)$ joining 0 with φ. The K-Energy does not depend on the path chosen. In fact there is the following well known formula for ν_ω where $O(1)$ denotes a quantity which is bounded on $P(X,\omega)$.

$$\nu_\omega(\varphi) = \int_X \log\left(\frac{{\omega_\varphi}^n}{\omega^n}\right)\frac{{\omega_\varphi}^n}{V} - \mu(I_\omega(\varphi) - J_\omega(\varphi)) + O(1)$$

$$J_\omega(\varphi) := \frac{1}{V}\int_X \sum_{i=0}^{n-1} \frac{\sqrt{-1}}{2\pi}\frac{i+1}{n+1}\partial\varphi \wedge \overline{\partial}\varphi \wedge \omega^i \wedge {\omega_\varphi}^{n-i-1}$$

$$I_\omega(\varphi) := \frac{1}{V}\int_X \varphi(\omega^n - {\omega_\varphi}^n).$$

We have written down the K-energy in the case when $\omega = c_1(X)$. Observe that ν_ω is essentially the *difference* of two positive terms. What is of interest for us is that the problem $(*)$ is not only a variational problem but a *minimization* problem. With this said we have the following fundamental result.

***Theorem* (S. Bando and T. Mabuchi [1]).** — *If $\omega = c_1(X)$ admits a Kähler Einstein metric then $\nu_\omega \geq 0$. The absolute minimum is taken on the solution to $(**)$ (which is unique up to automorphisms of X).*

Therefore a *necessary* condition for the existence of a Kähler Einstien metric is a bound from below on ν_ω. In order to get a *sufficient* condition one requires that the K-energy *grow* at a certain rate. Precisely, it is required that the K-Energy be *proper*. This concept was introduced by the second author in [**17**].

Definition 3. — *ν_ω is **proper** if there exists a strictly increasing function $f : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ (where $\lim_{T \longrightarrow \infty} f(T) = \infty$) such that $\nu_\omega(\varphi) \geq f(J_\omega(\varphi))$ for all $\varphi \in P(M, \omega)$.*

***Theorem* ([17]).** — *Assume that $Aut(X)$ is discrete. Then $\omega = c_1(X)$ admits a Kähler Einstein metric if and only if ν_ω is proper.*

The next result was established by the second author and Xiuxiong Chen. It holds in an *arbitrary* Kähler class ω. An alternative proof of this was given by Donaldson for polarized projective manifolds.

***Theorem* ([3]).** — *If ω admits a metric of csc then $\nu_\omega \geq 0$.*

In this paper our interest is to test for a lower bound of ν_ω along the large but finite dimensional group G of *matrices* in the polarized case. When we restrict our attention to G we make the connection with Mumfords' Geometric Invariant Theory. The past couple of years have witnessed quite a bit of activity on this problem due to this connection.

To put things in historical perspective consider the various formulations of the Futaki invariant.

i) 1983 Futaki ([**6**]) introduces his invariant as a lie algebra character on a Fano manifold X

$$F_\omega : \eta(X) \longrightarrow \mathbb{C}.$$

ii) 1986 Mabuchi (see [**10**]) integrates the Futaki invariant with the introduction of the K-energy map. The linearization of the K-energy along orbits of holomorphic vector fields is the real part of the Futaki invariant.

iii) 1992 Ding and Tian ([**4**]) introduced the *generalized* Futaki invariant. Here the *jumping of complex structures* is introduced. The limit of the derivative of the K-Energy map is identified with the generalized Futaki invariant of $X^{\lambda(0)}$ provided this limit has at most *normal* singularities.

iv) 1997 The CM polarization is defined (see [**17**]) for *smooth* families, as the relative canonical bundle is explicitly involved in the definition. K-Stability is defined in terms of special degenerations and the generalized Futaki invariant.