348 ASTÉRISQUE

2012

SÉMINAIRE BOURBAKI
VOLUME 2010/2011
EXPOSÉS 1027-1042

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Publié avec le concours du CENTRE NATIONAL DE LA RECHERCHE SCIENTIFIQUE

Séminaire BOURBAKI Avril 2011
63^{e} année, 2010-2011, n^{o} 1035, p. 233 à 263

THE FUNDAMENTAL LEMMA AND THE HITCHIN FIBRATION [after Ngô Bao Châu]

by **Thomas C. HALES**

The study of orbital integrals on p-adic groups has turned out to be singularly difficult.

(R. P. Langlands, 1992)

This report describes some remarkable identities of integrals that have been established by Ngô Bao Châu. My task will be to describe why these identities—collectively called the fundamental lemma (FL)—took nearly thirty years to prove, and why they have particular importance in the theory of automorphic representations.

1. BASIC CONCEPTS

1.1. Origins of the fundamental lemma (FL)

To orient ourselves, we give special examples of behavior that the theory is designed to explain.

EXAMPLE 1. — *We recall the definition of the holomorphic discrete series representations of $SL_2(\mathbb{R})$. For each natural number $n \geq 2$, let $V_{n,+}$ be the vector space of all holomorphic functions f on the upper half plane $\mathfrak{h}$ such that*

$$\int_{\mathfrak{h}} |f|^2 y^{n-2} dx\, dy < \infty.$$

$SL_2(\mathbb{R})$ acts on $V_{n,+}$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot f(z) = (-bz+d)^{-n} f\Big(\frac{az-c}{-bz+d}\Big).$$

Similarly, for each $n \geq 2$, there is an anti-holomorphic discrete series representation $V_{n,-}$. These infinite dimensional representations have characters that exist as locally *integrable functions* $\Theta_{n,\pm}$. *The characters are equal:* $\Theta_{n,+}(g) = \Theta_{n,-}(g)$, *except when* g *is conjugate to a rotation*

$$\gamma = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

When g *is conjugate to* γ, *a remarkable character identity holds:*

$$\Theta_{n,-}(\gamma) - \Theta_{n,+}(\gamma) = \frac{e^{i(n-1)\theta} + e^{-i(n-1)\theta}}{e^{i\theta} - e^{-i\theta}}. \tag{2}$$

It is striking that numerator of the difference of two characters of infinite dimensional representations collapses to the character of a two dimensional representation $\gamma \mapsto \gamma^{n-1}$ *of the group* H *of rotations. Shelstad gives general characters identities of this sort* [**49**].

We find another early glimpse of the theory in a letter to Singer from Langlands in 1974 [**33**]. Singer had expressed interest in a particular alternating sum of dimensions of spaces of cusp forms of $G = SL_2$ over a totally real number field F. Langlands's reply to Singer describes then unpublished joint work with Labesse [**32**]. Without going into details, we remark that in the calculation of this alternating sum, there is again a collapse in complexity from the three dimensional group SL_2 to a sum indexed by one-dimensional groups H (of norm 1 elements of totally imaginary quadratic extensions of F).

These two examples fit into a general framework that have now led to major results in the theory of automorphic representations and number theory, as described in Section 7. Langlands holds that methods should be developed that are adequate for the theory of automorphic representations in its full natural generality. This means going from SL_2 (or even a torus) to all reductive groups, from one local field to all local fields, from local fields to global fields and back again, from the geometric side of the trace formula to the spectral side and back again. Moreover, interconnections between different reductive groups and Galois groups should be included, as predicted by his general principle of functoriality.

Thus, from these early calculations of Labesse and Langlands, the general idea developed that one should account for alternating sums (or κ-sums as we shall call them because they occasionally involve roots of unity other than ± 1) that appear in the harmonic analysis on a reductive group G in terms of the harmonic analysis on groups H of smaller dimension. The FL is a concrete expression of this idea.

1.2. Orbital integrals

This section provides brief motivation about why researchers care about integrals over conjugacy classes in a reductive group. Further motivation is provided in Section 7.

It is a basic fact about the representation theory of a finite group that the set of irreducible characters is a basis of the vector space of class functions on the group. A second basis of that vector space is given by the set of characteristic functions of the conjugacy classes in the group. We will loosely speak of any linear relation among the set of characteristic functions of conjugacy classes and the set of irreducible characters as a *trace formula.*

More generally, we consider a reductive group G over a local field. Each admissible representation π of G defines a *distribution character:*

$$f \mapsto \operatorname{trace} \int_G f(g)\pi(g)\, dg, \quad f \in C_c^\infty(G),$$

with dg a Haar measure on G. A trace formula in this context should be a linear relation among characteristic functions of conjugacy classes and distribution characters. To put all terms of a trace formula on equal footing, the characteristic function of a conjugacy class must also be treated as a distribution, called an *orbital integral*:

$$f \mapsto \mathbf{O}(\gamma, f) = \int_{I_\gamma \backslash G} f(g^{-1}\gamma g)\, dg, \quad f \in C_c^\infty(G),$$

where I_γ is the centralizer of $\gamma \in G$.

The FL is a collection of identities among orbital integrals that may be used in a trace formula to obtain identities among representations π.

1.3. Stable conjugacy

At the root of these κ-sum formulas is the distinction between *ordinary conjugacy* and *stable conjugacy.*

EXAMPLE 3. — *A clockwise rotation and counterclockwise rotation*

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

in $SL_2(\mathbb{R})$ are conjugate by the complex matrix $\left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right)$, but they are not conjugate in the group $SL_2(\mathbb{R})$ when $\theta \notin \mathbb{Z}\pi$. Indeed, a matrix calculation shows that every element of $GL_2(\mathbb{R})$ that conjugates the rotation to counter-rotation has odd determinant, thereby falling outside $SL_2(\mathbb{R})$. Alternatively, they are not conjugate in $SL_2(\mathbb{R})$ because the character identity (2) separates them.

Let G be a reductive group defined over a field F with algebraic closure $\bar{F}$.

DEFINITION 4. — *An element $\gamma' \in G(F)$ is said to be stably conjugate to a given regular semisimple element $\gamma \in G(F)$ if γ' is conjugate to γ in the group $G(\bar{F})$.*

There is a Galois cohomology group that can be used to study the conjugacy classes within a given stable conjugacy class. Let I_γ be the centralizer of an element $\gamma \in G(F)$. The centralizer is a Cartan subgroup when γ is a (strongly) regular semisimple element. Write $\gamma' = g^{-1}\gamma g$, for $g \in G(\bar{F})$. For every element σ of the Galois group $\mathrm{Gal}(\bar{F}/F)$, we have $g\,\sigma(g)^{-1} \in I_\gamma(\bar{F})$. These elements define in the Galois cohomology group $H^1(F, I_\gamma)$ a class, which does not depend on the choice of g. It is the trivial class when γ' is conjugate to γ.

EXAMPLE 5. — *The centralizer I_γ of a regular rotation γ is the subgroup of all rotations in $SL_2(\mathbb{R})$. The group $I_\gamma(\mathbb{C})$ is isomorphic to $\mathbb{C}^\times$. Each cocycle is determined by the value $r \in I_\gamma(\mathbb{C}) = \mathbb{C}^\times$ of the cocycle on the generator of* $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$. *A given $r \in \mathbb{C}^\times$ satisfies the cocycle condition when $r \in \mathbb{R}^\times$ and represents the trivial class in cohomology when r is positive. This identifies the cohomology group:*

$$H^1(\mathbb{R}, I_\gamma) = \mathbb{R}^\times/\mathbb{R}_+^\times = \mathbb{Z}/2\mathbb{Z}.$$

This cyclic group of order two classifies the two conjugacy classes within the stable conjugacy class of a rotation.

When F is a local field, $A = H^1(F, I_\gamma)$ is a finite abelian group. Every function $A \to \mathbb{C}$ has a Fourier expansion as a linear combination of characters κ of A. The *theory of endoscopy* is the subject that studies stable conjugacy through the separate characters κ of A. Allowing ourselves to be deliberately vague for a moment, the idea of endoscopy is that the Fourier mode of κ (for given I_γ and G) produces oscillations that cause some of the roots of G to cancel away. The remaining roots are reinforced by the oscillations and become more pronounced. The root system consisting of the pronounced roots defines a group H of smaller dimension than G. With respect to the harmonic analysis on the two groups, the mode of κ on the group G should be related to the dominant mode on H.

1.4. Endoscopy

The smaller group H, formed from the "pronounced" subset of the roots of G, is called an *endoscopic group*. Hints about how to define H precisely come from various sources.

- It should be constructed from the data (G, I_γ, κ), with γ regular semisimple.
- Its roots should be a subset of the roots of G (although H need not be a subgroup of G).