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GROWTH OF A PRIMITIVE OF A DIFFERENTIAL FORM

by Jean-Claude Sikorav

Abstract. — For an exact differential form on a Riemannian manifold to have a primitive bounded by a given function f, by Stokes it has to satisfy some weighted isoperimetric inequality. We show the converse up to some constants if M has bounded geometry. For a volume form, it suffices to have the inequality ($|\Omega| \leq \int_{\partial\Omega} f \mathrm{d}\sigma$ for every compact domain $\Omega \subset M$). This implies in particular the "well-known" result that if M is the universal covering of a compact Riemannian manifold with non-amenable fundamental group, then the volume form has a bounded primitive. Thanks to a recent theorem of A. Żuk, we also obtain that if the fundamental group is infinite, the volume form always has a primitive with linear growth.

Résumé (*Croissance d'une primitive d'une forme différentiable*)

Pour qu'une forme différentielle exacte sur une variété riemannienne M ait une primitive majorée par une fonction f donnée, il faut d'après Stokes satisfaire une certaine inégalité isopérimétrique pondérée. Nous montrons une réciproque à des constantes près si la variété est à géométrie bornée. Pour une forme volume, l'inégalité ($|\Omega| \leq \int_{\partial\Omega} f \mathrm{d}\sigma$ pour tout domaine compact $\Omega \subset M$) suffit. Ceci implique en particulier le résultat « bien connu » que si M est le revêtement universel d'une variété riemannienne compacte à groupe fondamental non moyennable, la forme volume a une primitive bornée. Grâce à un théorème récent d'A. Żuk, nous obtenons aussi que si le groupe fondamental est infini, la forme volume a toujours une primitive à croisssance linéaire.

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1. Statement of the results and comments

Let M be a complete and non compact Riemannian manifold, $\omega \in \Omega^q(M)$ be an exact differential form of degree q, and $f : M \to \mathbb{R}_+$ be a continuous function. We want to find sufficient conditions for the existence of a primitive $\tau \in \Omega^{q-1}(M)$ such that $|\tau| \leq f$. Stokes' formula gives as a necessary condition the weighted isoperimetric inequality

$$(1) \qquad \Big| \int_T \omega \Big| \leq \int_{|\partial T|} f \quad \text{for every} \quad T \in \mathcal{S}^1_q(M).$$

Here $\mathcal{S}^1_q(M)$ denotes the vector space of singular q-chains $T = \sum \lambda_i s_i$ of class $\mathcal{C}^1$, and

$$\int_{|S|} f := \sum_i |\lambda_i| \int_{\Delta^q} (f \circ s_i)\, |\Lambda^q \mathrm{d}\sigma|.$$

Examples. — If M is simply connected and has nonpositive curvature, then any closed and bounded form has a primitive with at most linear growth, this being clearly optimal by Stokes in the case $M = \mathbb{R}^2$, $\omega = x\mathrm{d}y$. If the curvature is $\leq -a^2 < 0$, then the primitive is even bounded if $q \geq 2$.

On the other hand, there is an example of Gromov (see [G3], 3.K'$_3$, 6.B$_1$ (c)) for $q = 2$, M the universal covering of a compact X, and ω lifted from X, in which the inequality (1) implies that no primitive of ω has recursive growth!

Here we investigate the following

Question. — Assume that (1) holds. Does ω have a primitive $\tau \in \Omega^{q-1}(M)$ such that $|\tau| \leq f$? Or at least, such that $|\tau|_x \leq C_1 \max_{B(x,C_2)} f$?

The existence of a primitive such that $|\tau| \leq f$ follows from Hahn-Banach if we allow τ to be flat in the sense of Whitney [W] (roughly, this means that τ has measurable coefficients and $\mathrm{d}\tau = \omega$ holds in the sense of currents). To obtain a result for smooth forms, we shall assume that M has *bounded geometry* in the sense that it is complete, its sectional curvature is bounded in absolute value and its injectivity radius is bounded below. Examples include coverings of compact manifolds and leaves of foliations on compact manifolds. Such a manifold admits a triangulation with bounded geometry, in a sense made precise in section 2. Our main result is the

Theorem 1.1. — *Let M be a Riemannian manifold, with a triangulation K of bounded geometry. Let $\omega \in \Omega^q(M)$ be a closed q-form, and let $f \in C^0(M, \mathbb{R}_+)$ be such that*

$$(2) \qquad \Big| \int_T \omega \Big| \leq \int_{|\partial T|} f \quad \textit{for every simplicial chain } T \in C_q(K).$$

Then ω has a primitive τ such that, for some constants $C_1(M,K)$ and $C_2(M,K)$, one has

$$|\tau|_x \leq C_1 \max_{B(x,C_2)} \big(|\omega| + f\big).$$

I do not know if (assuming the stronger isoperimetric inequality (1)) one can dispense with the assumption of bounded geometry, or if one can drop $|\omega|$ in the estimate.

In the case of volume forms, we get:

Corollary 1.2. — *Let (M,K) be as above, with M oriented. Assume that*

$$|\Omega| \leq \int_{\partial\Omega} f \,\mathrm{d}\sigma \quad \textit{for every simplicial domain } \Omega \subset M. \tag{3}$$

Then the volume form ν has a primitive τ such that

$$|\tau|_x \leq C_1 \max_{B(x,C_2)} f.$$

Combining this with a recent result of A. Żuk [Z], we obtain:

Corollary 1.3. — *Let X be a compact oriented Riemannian manifold with infinite fundamental group. Then the volume form on the universal covering $M = \widetilde{X}$ has a primitive τ with at most linear growth.*

Comment. — To my knowledge, the first mention of growth of primitives was made by D. Sullivan in 1976 (see [Su]). He asked whether, on an oriented manifold satisfying the inequality $|\Omega| \leq \mathrm{Cst.\,vol}(\partial\Omega)$ for every compact domain $\Omega \subset M$, the volume form has a bounded primitive (M is "open at infinity"). He was especially interested in the case when M is a leaf of a foliation on a compact manifold. In the case when M is the universal cover of a compact manifold X, the isoperimetric inequality is equivalent to the Følner criterion for the non-amenability of $\pi_1(X)$ (see [GLP], chap. 6).

A positive answer to the question of Sullivan has been asserted (without any restrictions) by M. Gromov (see [G1], p. 197). R. Brooks (see [Br], pp. 61–62), sketches a proof "conceptually simple but with some unpleasant technical details": one first finds, for a suitable triangulation [geometrically bounded presumably], a bounded cochain such that $\mathrm{d}\psi = \mathrm{vol}$. Then one smooths out ψ after letting this triangulation get arbitrarily small.

Another proof (under the assumption of bounded geometry) has been given by J. Block and S. Weinberger (see [Bl-W], remark after Theorem 3.1, *cf.* also [A], Thm. 2.13), but it seems somewhat elliptic.

Another case which has had important applications in algebraic geometry is the following [G2]: if $M = \widetilde{X}$ where X is a compact manifold equipped with a Kähler form $\overline{\omega}$, then $(X, \overline{\omega})$ is said to be Kähler-hyperbolic if ω has a bounded primitive. Note that in all known examples the growth of the primitive is at most linear. In the symplectic case on the other hand, one can find an exponential growth by taking X to be a T^2-bundle on T^2 with hyperbolic monodromy.

Finally, in [G3], 5.B$_5$, Gromov investigates the general problem of growth of primitives of bounded forms, which he relates via Stokes to "cofilling inequalities". One can find there a wealth of related examples and questions, some of which we plan to tackle in a forthcoming paper.

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2. Triangulations of bounded geometry

A suitable version of the Cairns-Whitehead triangulation theorem implies that every Riemannian manifold with bounded geometry admits a smooth *triangulation with bounded geometry* (*cf.* [A], theorem 1.14) in the following sense:

(BG$_1$) the link of each simplex s contains at most S simplices, S independent of s;

(BG$_2$) each simplex is quasi-isometric to a standard simplex, *i.e.* there exists a diffeomorphism $\varphi_s : s \to \Delta^{\dim s}$ such that $|\mathrm{d}\varphi_s^{\pm 1}| \le L$, L independent of s.

We shall assume a slightly stronger version of (BG$_2$), easy to obtain by subdividing:

(BG$_3$) φ_s can be extended with the same property $|\mathrm{d}\varphi_s^{\pm 1}| \le L$ to a neighbourhood $U(s)$ of s in M, sending it to a fixed neighbourhood of $\Delta^{\dim s}$ in $\mathbb{R}^n$, $n = \dim M$.

Note that if M covers a compact X, then any smooth triangulation lifted from X has bounded geometry in this sense.

3. Proof of the theorem

Proceeding as in [So], we construct the primitive as F. Laudenbach in [L], who in turns follows the constructive proof of De Rham's theorem in [Sin-T], pp. 162–173. The new point is the introduction of explicit estimates at each step.

First step. — We reduce the theorem to the case when $\int_s \omega = 0$ for every $s \in K^{(q)}$.

1) Consider the simplicial cochain $I^q(\omega) \in C^q(K)$, image of ω by the integration morphism $I^q : \Omega^q(M) \to C^q(K)$. The hypothesis implies

$$\big|I^q(\omega)(T)\big| \leq V_{q-1} \|\partial T\|_f, \quad \forall T \in C_q(K),$$

where $V_{q-1} = \max_{s \in K^{(q)}} \mathrm{vol}(s)$ and $\| \sum \lambda_i s_i \|_f = \sum |\lambda_i| \max_{s_i} f$, seminorm on $C_{q-1}(K)$. By Hahn-Banach, we can define a linear form $t_\omega \in C^{q-1}(K)$ which satisfies

- $t_\omega(\partial T) = I^q(\omega)(T)$ for every $T \in K^{(q)}$, *i.e.* $\delta t_\omega = I^q(\omega)$;
- $|t_\omega(S)| \leq V_{q-1} \|S\|_f$ for every $S \in C_{q-1}(K)$.

In particular, we have

$$\big|t_\omega(s)\big| \leq V_{q-1} \max_s f \quad \forall s \in K^{(q-1)}.$$

2) Since K has bounded geometry, there exists a partition of unity $\{g_j\}$ subordinate to the covering $\{\mathrm{st}(v_j)\}$ (where (v_j) are the vertices of K), such that the differentials $|\mathrm{d}g_j|$ are bounded by a constant D. Here $\mathrm{st}(v)$ denotes the star of the vertex v, *i.e.* the union of all simplices containing v. Note that it is a neighbourhood of v which is sandwiched between two balls of fixed radii.

We can then construct a right inverse $P^* : C^*(K) \to \Omega^*(M)$ to I^*, commuting with the differentials (see [Sin-T], Step 2, p. 166):

$$P^q(t) = \sum_{s \in K^{(q)}} t(s) P^q(s^*),$$

where s^* is the generator of $C^q(K)$ dual to s (*i.e.* $s^*(\sigma) = \delta_{s,\sigma}$) and

$$P^q\big(\langle v_{j_0}, \cdots, v_{j_q}\rangle^*\big) = q\,! \sum_{i=0}^{q} g_{j_i}\, \mathrm{d}g_{j_0} \wedge \cdots \wedge \widehat{\mathrm{d}g_{j_i}} \wedge \cdots \wedge \mathrm{d}g_{j_q}.$$

It satisfies $\mathrm{supp}\, P^q(s^*) \subset \mathrm{st}(s)$ and $\|P^q(s^*)\|_{L^\infty} \leq (q+1)!\, D^q$. Thus, if $\mathrm{st}_q(x)$ is the set of q-simplices s such that $x \in \mathrm{st}(s)$, we get the estimate

$$\big|P^q(t)\big|_x \leq S(q+1)!\, D^q \max_{s \in \mathrm{st}_q(x)} \big|t(s)\big|.$$

Each simplex in $\mathrm{st}_q(x)$ is contained in $B(x, 2d)$ where $d = \max \mathrm{diam}\, s \leq L\sqrt{n}$. Thus for $t = t_\omega$ and $t = I^q(\omega)$, we obtain

$$\big|P^{q-1}(t_\omega)\big|_x \leq Sq\,!\, D^{q-1} V_{q-1} \max_{B(x,2d)} f,$$

$$\big|P^q I^q(\omega)\big|_x \leq S(q+1)!\, D^q V_q \max_{B(x,2d)} |\omega|.$$