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POSITIVITY OF QUADRATIC BASE CHANGE L-FUNCTIONS

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ABSTRACT. — We show that certain quadratic base change L-functions for Gl(2) are non-negative at their center of symmetry.

RÉSUMÉ (*Positivité des fonctions L du changement de base quadratique*)
On montre que certaines des fonctions L de Gl(2) obtenues par changement de base quadratique sont positives en leur centre de symétrie.

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1. The main theorem

Let E/F be a quadratic extension of number fields and $\eta_{E/F}$ or simply η the quadratic character of F attached to E, $\{1,\tau\}$ the Galois group of E/F. We will often write
$$\tau(z)=\overline{z}.$$
We will denote by U_1 the unitary group in one variable, that is, the group of elements of norm 1 in $E^\times$. Suppose that π is an automorphic cuspidal representation of $\mathrm{Gl}(2,F_{\mathbb{A}})$ whose central character ω is trivial on the group of norms. In other words $\omega=1$ or $\omega=\eta$. We assume that π is not dihedral with respect to E so that the base change representation Π of π to $\mathrm{Gl}(2,E)$ is still automorphic and cuspidal. Let Ω be an idele class character of E whose restriction to $F_{\mathbb{A}}^\times$ is equal to ω. Our main result is the following theorem:

Theorem 1. — *With the previous notations:* $L\big(\frac{1}{2},\Pi\otimes\Omega^{-1}\big)\geq 0$.

If $\omega=1$ and $\Omega=1$ then $L(s,\Pi)=L(s,\pi)L(s,\pi\otimes\eta)$ and the result has been established by Guo (see [G1], under some restrictions on E/F). As a matter of fact, by using results on averages of L-functions (see [FH]), Guo is able to prove that $L(\frac{1}{2},\pi)\geq 0$, which then implies our result for $\omega=1$, $\Omega=1$, without restriction on E/F. At any rate, Baruch and Mao [BM] have independently established that $L(\frac{1}{2},\pi)\geq 0$ if $\omega=1$. However, the present result–where Ω needs not be trivial–is more general, even in the case $\omega=1$.

Results on the positivity of $\mathrm{Gl}(2)$ L-functions have been considered by many mathematicians (see, for instance, [BFH], [Gr], [K], [Kk], [KS], [KZ], [S], [R], [S], [W3], [Ya]). Specially, the positivity of the twisted L-function at hand has been investigated (for holomorphic forms) in [GZ].

We note that $\Omega^\tau=\Omega^{-1}$ and Π is self-contragredient: $\widetilde{\Pi}=\Pi$. Thus
$$L(s,\Pi\otimes\Omega^{-1})=L\big(s,\Pi^\tau\otimes(\Omega^{-1})^\tau\big)=L(s,\Pi\otimes\Omega)=L(s,\widetilde{\Pi}\otimes\Omega).$$
Likewise,
$$\epsilon(s,\Pi\otimes\Omega^{-1})\epsilon(1-s,\Pi\otimes\Omega^{-1})=\epsilon(s,\Pi\otimes\Omega^{-1})\epsilon(1-s,\widetilde{\Pi}\otimes\Omega)=1.$$
In particular $\epsilon(\frac{1}{2},\Pi\otimes\Omega^{-1})=\pm1$. Thus, despite the fact that $\Pi\otimes\Omega^{-1}$ is not necessarily self-contragredient, the L-function $L(s,\Pi\otimes\Omega^{-1})$ is symmetric:
$$L(s,\Pi\otimes\Omega^{-1})=\epsilon(s,\Pi\otimes\Omega^{-1})L(1-s,\Pi\otimes\Omega^{-1}).$$

The following lemma is easily verified:

Lemma 1. — *Let v_0 be a place of F. If v_0 is inert and v is the corresponding place of E then:*
$$L\big(\tfrac{1}{2},\Pi_v\otimes\Omega_v^{-1}\big)>0.$$
If v_0 splits into v_1 and v_2 then:
$$L\big(\tfrac{1}{2},\Pi_{v_1}\otimes\Omega_{v_1}^{-1}\big)L\big(\tfrac{1}{2},\Pi_{v_2}\otimes\Omega_{v_2}^{-1}\big)>0.$$

Let S_0 be a finite set of places of F and S the corresponding set of places of E. Set
$$L^S(s,\Pi\otimes\Omega^{-1})=\prod_{v\notin S}L(s,\Pi_v\otimes\Omega_v^{-1}).$$
In view of the lemma, the statement of the theorem is equivalent to the positivity of $L^S(\frac{1}{2},\Pi\otimes\Omega^{-1})$.

If Π is dihedral with respect to E, then Π is associated with an idele class character Ξ of E whose restriction to $F_{\mathbb{A}}^{\times}$ is $\omega\eta$. Thus $\Xi^{\tau}=\Xi^{-1}$ and
$$L(s,\Pi\otimes\Omega^{-1})=L(s,\Xi\,\Omega^{-1})L(s,\Xi^{\tau}\Omega^{-1})=L(s,\Xi\,\Omega^{-1})L(s,\Xi^{-1}\Omega^{-1}).$$
If Ω is trivial or even quadratic this is ≥ 0. At any rate, in general, $\Xi\,\Omega^{-1}$ and $\Xi^{-1}\Omega^{-1}$ have η for restriction to $F_{\mathbb{A}}^{\times}$. Thus there are cuspidal representations π_1 and π_2 of $\mathrm{Gl}(2,F_{\mathbb{A}}^{\times})$ with trivial central character such that:
$$L(s,\Xi\,\Omega^{-1})=L(s,\pi_1),L(s,\Xi^{-1}\Omega^{-1})=L(s,\pi_2)$$
and by the results already quoted each factor is ≥ 0 at $s=\frac{1}{2}$. We will not discuss this case but remark that, by considering the discrete but non-cuspidal terms in our trace formula, we could probably handle this case as well.

The proof of the theorem is based on a careful analysis of the relative trace formula of [J2] (In the case $\Omega=1$ we could, like Guo, use the simpler trace formula of [J1].) Namely, we consider an inner form G of $\mathrm{Gl}(2,F)$ which contains a torus T isomorphic to $E^{\times}$. There is then an $\epsilon\in F^{\times}$, uniquely determined modulo $\mathrm{Norm}(E^{\times})$, such that the pair (G,T) is isomorphic to the pair (G_ϵ,T) defined as follows. We denote by $\mathbb{H}_\epsilon$ the semi-simple algebra of matrices $g\in M(2,E)$ of the form
$$g=\begin{pmatrix} a & \epsilon b\\ \overline{b} & \overline{a}\end{pmatrix} \tag{1}$$
and by G_ϵ its multiplicative group. Then
$$T=\Big\{t=\begin{pmatrix} a & 0\\ 0 & \overline{a}\end{pmatrix}\Big\}.$$
We let Z be the center of G_ϵ. We regard Ω as a character of $T(F_{\mathbb{A}})$: $t\mapsto\Omega(a)$. Suppose that f is a smooth function of compact support on $G_\epsilon(F_{\mathbb{A}})$. We form as usual a kernel
$$K_f(x,y):=\int_{Z(F_{\mathbb{A}})/F^{\times}}\sum_{\xi\in G_\epsilon(F)}f(x^{-1}z\,\xi y)\,\omega(z)\,\mathrm{d}z$$
and a distribution
$$J_\epsilon(f):=\int_{(Z(F_{\mathbb{A}})T(F)\backslash T(F_{\mathbb{A}}))^2}K_f(t_1,t_2)\,\Omega(t_1)^{-1}\mathrm{d}t_1\,\Omega(t_2)\mathrm{d}t_2.$$

We have a spectral decomposition of the kernel:

$$K_f = \sum_\sigma K_{f,\sigma} + K_{f,\mathrm{cont}}\,;$$

the sum on the right is over all irreducible (cuspidal) automorphic representations σ of G_ϵ: if G_ϵ is not split, by cuspidal we mean an irreducible automorphic representation which is not one-dimensional. The term $K_{f,\mathrm{cont}}$ represents the contribution of the one dimensional representations and the continuous spectrum which is present only if G_ϵ is split, that is, ϵ is a norm. For every σ the kernel $K_{f,\sigma}$ is defined by

$$K_{f,\sigma}(x,y) = \sum_\phi \rho(f)\phi(x)\overline{\phi(y)},$$

the sum over an orthonormal basis of the space of σ. We define then:

$$J_\sigma(f) := \int_{(Z(F_\mathbb{A})T(F)\backslash T(F_\mathbb{A}))^2} K_{f,\sigma}(t_1,t_2)\,\Omega(t_1)^{-1}\mathrm{d}t_1\,\Omega(t_2)\mathrm{d}t_2.$$

This is a distribution of positive type: if $f = f_1 * f_1^*$ where $f_1^*(g) := \overline{f}_1(g^{-1})$ then

$$J_\sigma(f) = \sum \nu\big(\rho(f_1)\phi\big)\,\overline{\nu\big(\rho(f_1)\phi\big)},$$

where we have set

(2) $$\nu(\phi) := \int_{Z(F_\mathbb{A})T(F)\backslash T(F_\mathbb{A})} \phi(t)\Omega(t)^{-1}\,\mathrm{d}t;$$

thus $J_\sigma(f) \geq 0$. Moreover, if ν is not identically zero on the space of σ, or as we shall say, if σ is *distinguished* by (T,Ω), then every local component σ_{v_0} is *distinguished* by (T_{v_0},Ω_{v_0}), that is, admits a non-zero continuous linear form ν_{v_0} such that $\nu_{v_0}(\pi_{v_0}(t)u) = \Omega_{v_0}(t)\nu_{v_0}(u)$ for all $t \in T_{v_0}$ and all smooth vectors u. The dimension of the space of such linear forms is one. One can then define a local distribution

$$J_{\sigma_{v_0}}(f_{v_0}) = \sum \nu_{v_0}\big(\rho(f_{v_0})u\big)\,\overline{\nu_{v_0}(u)},$$

the sum over an orthonormal basis. The distribution $J_{\sigma_{v_0}}$ is defined within a positive factor. It is of positive type. Normalizing in an appropriate way we get

(3) $$J_\sigma(f) = C(\sigma)\prod_{v_0} J_{v_0}(f_{v_0}),$$

where the constant $C(\sigma)$ is positive. Assuming that $L(\frac12,\Pi\otimes\Omega^{-1}) \neq 0$ we can find an ϵ such that there is an automorphic representation σ of G_ϵ corresponding to π and distinguished by (T,Ω) (see [J2], [W4]). Another goal of the paper is to obtain an *explicit* decomposition of the above form, with a specific normalization (Theorem 2). The crux of the matter is then to show that $C(\sigma)$ is essentially equal to $L(\frac12,\Pi\otimes\Omega^{-1})$ which gives the positivity result. Possibly, this can be used to provide lower bounds for $L(\frac12,\Pi\otimes\Omega^{-1})$.

We note that if ϵ exists then it is unique. Indeed, this follows at once from the following local fact: if v_0 is inert, ϵ not a norm at v_0, π is a square integrable representation of $\mathrm{Gl}(2, F_{v_0})$, and σ the representation of $G_\epsilon(F_{v_0})$ corresponding to π then π and σ cannot be both distinguished by (T_{v_0}, Ω_{v_0}) (see [W4]).

We stress that there is no direct way to compute the constant $C(\sigma)$ because there is no direct relation between the global linear form ν and the local linear forms ν_{v_0}. The situation at hand (a globally defined distribution of positive type decomposed as a product over all places of F of local distributions of positive type, times the appropriate special values of L-functions) is, conjecturally, quite general. In this situation, the positivity of the special value of the L-function follows. One can view this question as a generalization of the problem of computing the Tamagawa number. This is our motivation for investigating in detail the present situation.

We proceed as follows. We introduce the matrices

$$w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad w_\epsilon = \begin{pmatrix} 0 & \epsilon \\ 1 & 0 \end{pmatrix}. \tag{4}$$

It will be more convenient to consider instead the distributions

$$\theta_\epsilon(f) := \int_{(Z(F_{\mathbb{A}})T(F)\backslash T(F_{\mathbb{A}}))^2} K_f(t_1, t_2)\, \Omega(t_1 t_2)^{-1} \mathrm{d}t_1 \mathrm{d}t_2,$$

and, for σ an automorphic representation of G_ϵ,

$$\theta_\sigma(f) := \int_{(Z(F_{\mathbb{A}})T(F)\backslash T(F_{\mathbb{A}})^2} K_{f,\sigma}(t_1, t_2)\, \Omega(t_1 t_2)^{-1} \mathrm{d}t_1 \mathrm{d}t_2.$$

Thus

$$J_\sigma(f) = \theta_\sigma\big(\rho(w_\epsilon) f\big)$$

and likewise for J_ϵ. We will decompose explicitly θ_σ into a product over all places v_0 of F of local distributions $\theta_{\sigma_{v_0}}$.

To that end, we compute the geometric expression for $\theta_\epsilon(f)$. A set of representatives for the double cosets of $T(F)$ in $G_\epsilon(F)$ is given by the matrices:

$$\mathbf{1}_2, \quad \begin{pmatrix} 0 & \epsilon \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & \epsilon\,\beta \\ \overline{\beta} & 1 \end{pmatrix}, \quad \beta \in E^\times / U_1(F).$$

We define orbital integrals. For $\xi \neq 1$ in $\mathrm{Norm}(E^\times)\epsilon$ we write $\xi = \beta\overline{\beta}\epsilon$ and set:

$$H(\xi; f) = \iint f\Big[t_1 \begin{pmatrix} \beta^{-1} & \epsilon \\ 1 & \overline{\beta}^{-1} \end{pmatrix} t_2\Big] \Omega(t_1 t_2^{-1}) \mathrm{d}t_1 \mathrm{d}t_2. \tag{5}$$

Note that the right hand side of the integral depends only on $\beta\overline{\beta}\epsilon$, which justifies the notations. In addition, we define

$$H(\infty; f) := \int f\Big[t_1 \begin{pmatrix} 0 & \epsilon \\ 1 & 0 \end{pmatrix}\Big] \Omega(t_1) \mathrm{d}t_1, \tag{6}$$