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TOWARDS A MORI THEORY ON COMPACT KÄHLER THREEFOLDS III

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ABSTRACT. — Based on the results of the first two parts to this paper, we prove that the canonical bundle of a minimal Kähler threefold (*i.e.* K_X is nef) is good, *i.e.* its Kodaira dimension equals the numerical Kodaira dimension, (in particular some multiple of K_X is generated by global sections); unless X is simple. "Simple" means that there is no compact subvariety through the very general point of X and X not Kummer. Moreover we show that a compact Kähler threefold with only terminal singularities whose canonical bundle is not nef, admits a contraction unless X is simple with Kodaira dimension $-\infty$.

RÉSUMÉ (*À propos d'une théorie de Mori sur les variétés compactes kählériennes de dimension* 3*, III*)

Utilisant les résultats de la première et de la deuxième partie de ce travail, nous considérons des variétiés kählériennes minimales X de dimension 3, *i.e.* dont le fibré canonique K_X est nef. Alors K_X est un fibré « good », *i.e.* dont la dimension de Kodaira est égale à la dimension de Kodaira numérique, sous l'exception possible que X est simple, (*i.e.* il n'existe pas une sous-variété compacte contenant un points très general) et X non Kummer. Le deuxième théorème dit que les variétés kählériennes X de dimension 3 avec des singularités terminales de sorte que K_X n'est pas nef, ont des contractions de Mori.

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Introduction

In this note we continue the study of the bimeromorphic geometry of compact Kähler threefolds. The final aim should be – as in the algebraic case – to construct minimal models for threefolds with non-negative Kodaira dimension, describe the way how to get a minimal model, to prove abundance for the minimal models, *i.e.* semi-ampleness of the canonical bundle, and finally to construction Fano fibrations on appropriate models of threefolds with $\kappa = -\infty$.

Concerning abundance we show that the canonical divisor of a minimal Kähler threefold X is good, *i.e.* $\kappa(X) = \nu(X)$, where $\nu(X)$ denotes the numerical Kodaira dimension, *i.e.* the largest number m such that $K_X^m \not\equiv 0$; unless X is simple and non-Kummer. Here X is *simple*, if there is no positive-dimensional subvariety through the very general point of X and X is *Kummer* if it has a bimeromorphic model which is the quotient of a torus by a finite group. These simple non-Kummer varieties are expected not to exist but this can be only a consequence of a completely developped minimal model theory in the Kähler case. By [22], it follows from $\kappa(X) = \nu(X)$ that K_X is semi-ample, *i.e.* some multiple mK_X is generated by global sections. So abundance holds on Kähler threefolds with the possible exception of simple non-Kummer threefolds.

Furthermore we prove, using essentially Part 1 [1] and Part 2 [23] to this paper that a smooth compact Kähler threefold X with K_X not nef carries a contraction unless possibly X is simple and non-Kummer. The main steps in the proof are the following:

1) Construction of some curve $C \subset X$ with $K_X \cdot C < 0$. We distinguish the case $\kappa(X) \geq 0$ and $\kappa(X) = -\infty$. In the first case we examine carefully a member in the linear system $|mK_X|$ to construct C, in the second we use a result of a recent joint paper of Campana and the author saying that X is uniruled unless X is simple. In that sedond case it is immediately clear that we can choose C rational.

2) Next we make C rational (this step works for all compact Kähler threefolds). Here we construct from C a non-splitting family of irreducible curve and examine its structure. The reason why this family exists is the deformation lemma of Ein-Kollár: a curve C in a smooth threefold with $K_X \cdot C < 0$ moves.

3) The last step is the construction of a contraction from a rational curve C with $K_X \cdot C < 0$. This was in large parts already done in [1] and [23]; here we finish the study in sect. 1 of this paper.

We summarise the results of this paper in the following two theorems.

THEOREM 1. — *Let X be a minimal Kähler threefold ($\mathbb{Q}$-factorial with at most terminal singularities). Assume that X is not both simple and non-Kummer. Then $\kappa(X) = \nu(X)$, hence K_X is semi-ample. In particular, if $\kappa(X) = 0$*

(*and* X *not simple non-Kummer*), *then* $K_X \equiv 0$ *and* $mK_X = \mathcal{O}_X$ *for some positive* m.

Theorem 2. — *Let* X *be a smooth compact Kähler threefold with* K_X *not nef. Then* X *carries a contraction unless* (*possibly*) X *is simple with* $\kappa(X) = -\infty$.

By a contraction we mean a surjective map $\varphi : X \to Y$ to a normal compact variety with connected fibers such that $-K_X$ is φ-ample and $b_2(X) = b_2(Y)+1$. We would like to have that Y is again a Kähler space but at the moment we are still in trouble if φ is the blow-up of a smooth curve in the smooth threefold Y – in that case Y will be only Kähler if φ is choosen appropriately, namely the ray generated by the curves contracted by φ must be extremal in the dual cone to the Kähler cone of X (see [23]).

There are several problems arising.

(a) First of all we would like to contruct a curve with $K_X \cdot C < 0$ also in the "simple" case if K_X is not nef. This would mean that we should construct "directly" – *i.e.* without using any specific information on X some curve C with $K_X \cdot C < 0$. This requires certainly new techniques and probably a threefold proof would work in any dimension and also with terminal singularities.

(b) We need to prove the existence of contractions also for $\mathbb{Q}$-Gorenstein threefolds with terminal singularities in order to perform the Mori program. The Gorenstein case will probably be the same as the smooth case, but in the presence of non-Gorenstein singularities there obstructions to moving curves so that some new arguments are needed.

(c) We must overcome the difficulty with the Kähler property of Y in case of a blow-up $\varphi : X \to Y$ along a smooth curve.

As already observed in [23], one major consequence of this programme would be that simple Kähler threefolds are Kummer, in particular there are no simple threefolds of negative Kodaira dimension, all those being uniruled.

In the appendix we prove that Kummer threefolds T/G with algebraic dimension 0 have Kodaira dimension 0. This was already used in [23] and [2] but no proof was given.

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Preliminaries

0.1. — Let X be a compact complex manifold and L a line bundle on X. Fix a positive $(1,1)$-form ω on X. Then L is *nef* if for every $\epsilon > 0$ there exists a hermitian metric h_ϵ on L with curvature $\Theta_{h_\epsilon} \geq -\epsilon\,\omega$. Since X is compact, this notion does not depend on the choice of ω. If X is projective, we obtain the "old" definition, that $L \cdot C \geq 0$ for all irreducible compact curves $C \subset X$. For details we refer to [4].

Let X be an irreducible normal compact complex in class $\mathcal{C}$, *e.g.* a normal compact Kähler space. Let L be a line bundle on X. Then L is nef if there is a desingularisation $\pi : \hat{X} \to X$ such that $\pi^*(L)$ is nef. By [23, 4.6] this is independent on the choice of π at least in dimension 3.

0.2. — A normal compact Kähler threefold (n-fold) is minimal if X is $\mathbb{Q}$-factorial, has only terminal singularities and K_X (*i.e.* some mK_X) is nef.

0.3. — We will often use $C_{n,m}$ for Kähler threefolds: if $f : X \to Y$ is a surjective fiber space with X a smooth compact Kähler threefold (so $n = 3$) then $\kappa(X) \geq \kappa(Y) + \kappa(F)$, where F denotes the general fiber ($m = \dim Y$). See [7], [10], [27].

0.4. — A compact manifold is *simple* if there is no positive dimensional subvarieties through the very general point of X. The only known Kähler examples arise from tori. To make this more precise one says that a compact manifold (or normal compact space) is *Kummer* if X is bimeromorphic to a quotient T/G of a torus by a finite group G. The conjecture is that simple Kähler manifolds are Kummer. A standard reference here is [6].

0.5. — Some further notations: $a(X)$ denotes always the algebraic dimension of X. The irregularity of X is $q(X) = \dim H^1(X, \mathcal{O}_X)$. Finally $N_1(X) \subset H_2(X, \mathbb{R})$ is the vector space generated by the classes of irreducible curve. Inside $N_1(X)$ we have the closed cone $\overline{NE}(X)$ generated by the classes of irreducible curves.

1. Abundance for Kähler threefolds

In this section we prove the Abundance Conjecture for (non-simple) Kähler threefolds. First we show

THEOREM 1.1. — *Let X be a minimal Kähler threefold with $\kappa(X) = 0$. Assume that X is not both simple and non-Kummer. Then $K_X \equiv 0$.*

Proof. — The assertion being known for projective minimal threefolds by Miyaoka [19] and Kawamata [13], we shall assume that X is non-projective. Then by [2] X has a bimeromorphic model X' with at most quotient singularities such that there is a finite cover $Y \to X'$, étale in codimension 1, with Y a torus or a product $E \times S$ of an elliptic curve E with a K3-surface S. In the algebraic case such a conclusion is of course false: Calabi-Yau threefolds are simply connected. The reason why the non-algebraic case is somehow more special than the projective is the existence of holomorphic 2-forms on non-algebraic threefolds (Kodaira's theorem).

Now back in our specific situation, we conclude that $K_{X'} \equiv 0$. We claim that X' has only canonical singularities (a priori we know only that $(X', 0)$ has log terminal singularities). This is seen as follows. Choose $m > 0$ such that $mK_{X'} \simeq \mathcal{O}_{X'}$. Let $\pi : \hat{X} \to X'$ be a desingularisation. Write

$$mK_{\hat{X}} = \pi^*(mK_{X'}) + \sum a_i E_i$$

with E_i the exceptional components of π. We need to show that $a_i \geq 0$ for all i. If some $a_i < 0$, then $\pi_*(mK_{\hat{X}})$ is a proper subsheaf of $mK_{X'} = \mathcal{O}_{X'}$; hence $H^0(mK_{\hat{X}}) = 0$. Passing to a high multiple of m, we deduce $\kappa(\hat{X}) = -\infty$, a contradiction.

So X' has only canonical singularities. Take a partial crepant resolution $\pi : \tilde{X} \to X'$ [24]. One can even assure that $\tilde{X}$ is $\mathbb{Q}$-factorial [25], [12]. So $\tilde{X}$ has only terminal singularities and still $K_{\tilde{X}} \equiv 0$. Now the bimeromorphic meromorphic map $X \rightharpoonup \tilde{X}$ is an isomorphism in codimension 1 [8], [16]. Here we use the fact that K_X is nef! Thus $K_X \equiv 0$. □

Remark 1.2. — In case $q(X) > 0$, it is much easier to conclude; the argument being independent of [2]. Here is the reasoning in that case. Let $\alpha : X \to A$ be the Albanese map of X. Since $\kappa(X) = 0$, α has to be surjective (by $C_{3,1}$ and $C_{3,2}$). Let $f : X \to Y$ be the Stein factorisation of α (we shall see that actually α has connected fibers). Notice that $\kappa(F) = 0$ for the general fiber F of f.

1) Suppose $\dim A = \dim Y = 3$. Then $X \to A$ is unramified in codimension 1, in fact, otherwise K_X would contain the ramification divisor R whose image in A is a nef divisor with $\kappa > 0$, so that $\kappa(R) > 0$, hence $\kappa(X) > 0$. Of course, this is well-known, at least in the algebraic case, see [14]. So $\alpha = f$ (by the universal property of α), hence α is birational. Therefore

$$K_X = \alpha^*(K_A) + \sum_{i\in I} \lambda_i E_i = \sum \lambda_i E_i,$$

where E_i are the exceptional components of α and $\lambda_i > 0$. Then K_X being nef, it follows easily that $I = \varnothing$. So $K_X = \mathcal{O}_X$.

2) Next suppose that $\dim Y = 1$. Hence $Y = A$ is an elliptic curve and again $f = \alpha$. Since K_F is nef and $\kappa(F) = 0$, we deduce that K_F is torsion. Choose $m > 0$ such that mK_X is Cartier and that $h^0(mK_X) = 1$. In particular $mK_F = \mathcal{O}_F$. Let $s \in H^0(mK_X)$ be a non-zero section. Then $s|F = 0$ or $s|F$ has no zeroes. Writing

$$mK_X = \sum a_i D_i$$

with $a_i > 0$, we conclude that $D_i \cap F = \varnothing$, *i.e.* $\dim D_i = 0$. On the other hand $\sum a_i D_i$ is α-nef, and this is only possible if some multiple kmK_X is a multiple of fibers so that

$$kmK_X = \alpha^*(L)$$