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PROPERTIES OF WIENER-WINTNER DYNAMICAL SYSTEMS

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ABSTRACT. — In this paper we prove the following results. First, we show the existence of Wiener-Wintner dynamical system with continuous singular spectrum in the orthocomplement of their respective Kronecker factors. The second result states that if $f \in L^p$, p large enough, is a Wiener-Wintner function then, for all $\gamma \in (1+\frac{1}{2p}-\frac{\beta}{2}, 1]$, there exists a set X_f of full measure for which the series $\sum_{n=1}^{\infty} \frac{f(T^n x)e^{2\pi i n\varepsilon}}{n^\gamma}$ converges uniformly with respect to ε.

RÉSUMÉ (*Propriétés des systèmes dynamiques de Wiener-Wintner*)
Dans cette note nous démontrons les résultats suivants. Tout d'abord nous montrons l'existence de systèmes dynamiques ergodiques du type Wiener Wintner ayant un spectre singulier continu dans l'orthogonal de leur facteurs de Kronecker.Ensuite nous montrons que si $f \in L^p$ est une fonction du type Wiener-Wintner alors, pour $\gamma \in (1+\frac{1}{2p}-\frac{\beta}{2}, 1]$ on peut trouver un ensemble X_f de mesure pleine pour lequel la série $\sum_{n=1}^{\infty} \frac{f(T^n x)e^{2\pi i n\varepsilon}}{n^\gamma}$ converge uniformément en ε.

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Introduction

Let $(X, \mathcal{B}, \mu, T)$ be an ergodic dynamical system. Throughout this paper $\mathcal{K}$ will denote the Kronecker factor of T, i.e. the closed linear span in L^2 of the eigenfunctions for T. Wiener-Wintner functions and Wiener-Wintner dynamical systems were introduced and studied in [1] and [2].

DEFINITION 1. — *Let $(X, \mathcal{B}, \mu, T)$ be an ergodic dynamical system. A function f is a Wiener-Wintner function of power type β in L^1 if there exist finite positive constants C_f and β such that*

$$\left\| \sup_{\varepsilon} \left| \frac{1}{N} \sum_{n=1}^{N} f(T^n x) e^{2\pi i n \varepsilon} \right| \right\|_1 \leq \frac{C_f}{N^{\beta}}$$

for all positive integers N.

DEFINITION 2. — *An ergodic dynamical system $(X, \mathcal{B}, \mu, T)$ is a Wiener-Wintner dynamical system of power type β in L^1 if there exists in $\mathcal{K}^{\perp}$ a dense set (for the L^2 norm) of Wiener-Wintner functions of power type β in L^1.*

In this paper whenever we say Wiener-Wintner we mean Wiener-Wintner of power type β in L^1.

Among other properties these dynamical systems provide simpler proof of J. Bourgain [3] a.e. double recurrence result which answered a question of H. Furstenberg [4]. It was shown in [1] that K automorphisms, product of K automorphism with any other Wiener-Wintner dynamical system, and factors of Wiener-Wintner dynamical systems are Wiener-Wintner dynamical systems. Discrete spectrum transformations are trivially Wiener-Wintner dynamical systems. It was also shown that for almost all irrational α (with respect to Lebesgue measure) the skew products $(x, y) \to (x+\alpha, y+x)$ on the 2-Torus, $\mathbb{T}^2$, are Wiener-Wintner dynamical systems. This showed that there exist nontrivial Wiener-Wintner dynamical systems with zero entropy. All of the dynamical systems mentioned above have in fact a dense set of L^{∞} Wiener-Wintner functions. Note also that they all have Lebesgue spectrum in the orthocomplement of the Kronecker factor. This raises the question of finding an example of Wiener-Wintner dynamical systems with continuous singular spectrum in the orthocomplement of its Kronecker factor. We will show in this paper that the skew products $(x, y) \to (x+\alpha, y+\beta\{x\})$, α with unbounded partial quotients and β irrational, on $\mathbb{T}^2$, provide us with examples of Wiener-Wintner dynamical systems with continuous singular spectrum in the othocomplement of the Kronecker factor. Hence, the Wiener-Wintner property of a dynamical system is not characterized by the nature of its spectrum in $\mathcal{K}^{\perp}$, (singular or Lebesgue).

The first author also showed that if f is a Wiener-Wintner function, then there exists a set X_f of full measure for which the rotated one sided ergodic

Hilbert transform for f, i.e. the series $\sum_{n=1}^{\infty} \dfrac{f(T^n x)e^{2\pi in\varepsilon}}{n}$, converges for all ε. Moreover, for $x \in X_f$ the map

$$\varepsilon \longmapsto \sum_{n=1}^{\infty} \frac{f(T^n x)e^{2\pi in\varepsilon}}{n}$$

is continuous ([1]). In this paper we will study the convergence of the series $\sum_{n=1}^{\infty} \dfrac{f(T^n x)e^{2\pi in\varepsilon}}{n^{\gamma}}$ for $f \in L^p$ and $0 < \gamma < 1$. To get convergence and continuity in this case we will prove and use the property that

$$\lim_{N\to\infty} \frac{1}{N^{\gamma\delta}} \sum_{n=[N^{\delta}]}^{[(N+1)^{\delta}]} |f|(T^n x) = 0 \text{ a.e.} \quad \text{for } f \in L^p,\ 1 \le p \le \infty,$$

and for δ that depends on p. M. Schwartz showed that when $\gamma = \frac{\delta-1}{\delta}$ the result is not true. The averages $\frac{1}{N^{\delta-1}} \sum_{n=N^{\delta}}^{(N+1)^{\delta}} f(T^n x)$ will not converge a.e. even for characteristic functions ([10]). We show that for $\gamma > 1 + \frac{1}{p} - \frac{1}{\delta} > 1 - \frac{1}{\delta} = \frac{\delta-1}{\delta}$,

$$\frac{1}{N^{\gamma\delta}} \sum_{n=N^{\delta}}^{(N+1)^{\delta}} |f|(T^n x)$$

converges to 0. Note that convergence to zero is trivial for $f \in L^{\infty}$. In the case of L^{∞} i.i.d random variables it is easy to see that there exist Wiener-Wintner functions, namely the Rademacher functions, for which the map $\varepsilon \mapsto \sum_{n=1}^{\infty} \dfrac{f(T^n x)e^{2\pi in\varepsilon}}{n^{\gamma}}$ is not continuous if $\gamma \le 1/2$.

Throughout this paper references about Diophantine approximations can be found in the classical book of A. Khinchin on continued fractions ([7]) or in [8].

1. Existence of Wiener-Wintner dynamical systems with continuous singular spectrum in $\mathcal{K}^{\perp}$

Let α be irrational and β real. Let $T_{\alpha,\beta} : \mathbb{T}^2 \to \mathbb{T}^2$ be the skew product defined on the 2-torus by

$$T_{\alpha,\beta}(x, y) = (x + \alpha, y + \beta\{x\}) \mod 1$$

where $\{x\}$ is the fractional part of x. i.e. $\{x\} = x \mod 1$.

The case $\beta = 1$ was studied in [1]. It was shown that for almost all irrational α the system $(\mathbb{T}^2, \mathcal{B}(\mathbb{T}^2), T_{\alpha,1}, m)$ is a Wiener-Wintner dynamical system. These dynamical systems have Lebesgue spectrum in the orthocomplement of the Kronecker factor.

We will show that for all β irrational there exists a set $\mathcal{I}_{\beta}$ of α with unbounded partial quotients of full measure such that for all α in $\mathcal{I}_{\beta}$ the system

$(\mathbb{T}^2, \mathcal{B}(\mathbb{T}^2), T_{\alpha,\beta}, m)$ is a Wiener-Wintner dynamical system. In particular, the functions $f_{p,q}(x,y) = e^{2\pi ipx}e^{2\pi iqy}$, $q \neq 0$, form a dense set in $\mathcal{K}^\perp$ of Wiener-Wintner functions. Such systems were shown in [9] to have continuous singular spectrum in the orthocomplement of the Kronecker factor. The results in [9] have been extended by A. Iwanik, M. Lemanczyk, and C. Mauduit in [5].

THEOREM 1. — *For all $\beta \neq 0$ there exists a set I_β of irrational α of full measure such that for all α in I_β and for all $p, q (q \neq 0)$, we can find a constant $C_{q,\alpha,\beta}$ such that*

$$\sup_{\varepsilon} \left| \frac{1}{N} \sum_{n=1}^{N} f_{p,q}(T^n_{\alpha,\beta}(x,y)) e^{2\pi in\varepsilon} \right| \leq \frac{C_{q,\alpha,\beta}}{N^s}$$

for some $s > 0$, for all (x, y) and for all positive integers N.

For the proof of this theorem we will use the following lemmas:

LEMMA 1. — *For all integers m, $m \neq 0$, and for all real numbers β, $\beta \neq 0$*

$$\int_0^1 e^{2\pi i\beta n\{m\alpha\}} d\alpha = \frac{1}{2\pi i\beta n}(e^{2\pi i\beta n} - 1).$$

The proof of the above lemma is straightforward.

The following lemma can be extracted from the proof of the Proposition in [1].

LEMMA 2. — *Let $\{f_n\}$ be a sequence of uniformly bounded functions. If*

$$\int_0^1 \left| \frac{1}{N} \sum_{n=1}^{N} f_n(\alpha) \right| d\alpha \leq \frac{C}{N^r} \quad \textit{for some } r > 0,$$

then for almost every α,

$$\left| \frac{1}{N} \sum_{n=1}^{N} f_n(\alpha) \right| \leq \frac{C_{r,\alpha}}{N^{\rho_r}},$$

where $C_{r,\alpha}$ is a new positive constant that depends on r and α, and ρ_r is a positive constant less than r that depends on r.

Proof. — Without loss of generality, we can assume that $\{f_n\}$ is uniformly bounded by 1.
We will first prove the result for a subsequence, and then for the sequence itself.
Let $0 < s < r$. Then,

$$\int_0^1 \left[N^s \left| \frac{1}{N} \sum_{n=1}^{N} f_n(\alpha) \right| \right] d\alpha \leq \frac{C}{N^{r-s}}.$$

Let $N = M^l$ for l large enough. That is, $l(r-s) > 1$. This implies that

$$\sum_{M=1}^{\infty} \int \left|(M^l)^s \frac{1}{M^l} \sum_{n=1}^{M^l} f_n(\alpha)\right| d\alpha \le \sum_{M=1}^{\infty} \frac{C}{M^{l(r-s)}} < \infty.$$

Hence, for a.e. α

$$\left|\frac{1}{M^l} \sum_{n=1}^{M^l} f_n(\alpha)\right| \le \frac{C_\alpha}{(M^l)^s}.$$

Now, let $M^l \le N \le (M+1)^l$. Then,

$$\left|\frac{1}{N} \sum_{n=1}^{N} f_n(\alpha)\right| \le \left|\frac{1}{M^l} \sum_{n=1}^{M^l} f_n(\alpha)\right| + \frac{1}{M^l} \sum_{n=M^l+1}^{(M+1)^l} |f_n(\alpha)|.$$

The first term is less than or equal to $C_\alpha/(M^l)^s$ for a.e. α. The second term

$$\frac{1}{M^l} \sum_{n=M^l+1}^{(M+1)^l} |f_n(\alpha)| \le \frac{(M+1)^l - M^l}{M^l} \text{ since } |f_n(\alpha)| \le 1 \text{ for all } \alpha \text{ and for all } n$$

$$\le \frac{CM^{l-1}}{M^l} = \frac{C}{M}.$$

Hence, for a.e. α,

$$\left|\frac{1}{N} \sum_{n=1}^{N} f_n(\alpha)\right| \le \frac{C_\alpha}{M^{ls}} + \frac{C}{M} \le \frac{C_\alpha}{M^\gamma},$$

where $\gamma = \min\{ls, 1\}$. Note that

$$\frac{1}{M^\gamma} = \left(\frac{M+1}{M}\right)^\gamma \frac{1}{(M+1)^\gamma} \le 2^\gamma \frac{1}{(M+1)^\gamma} \le \frac{2^\gamma}{N^{\gamma/l}}.$$

Letting $\rho_r = \gamma/l$ we have the desired result. □

LEMMA 3. — *Let $f_h(x) : [0,1) \to S^1 = \{z \in \mathbb{C} : |z| = 1\}$ be the function defined by*

$$f_h(x) = e^{2\pi i q\beta(\{x\}+\{x+\alpha\}+\cdots+\{x+(h-1)\alpha\})}$$

where α is a fixed irrational number and $\beta \in \mathbb{R} \setminus \{0\}$. Then f_h has bounded variation, $V(f_h)$, and $V(f_h) = O(h)$.

Proof. — This follows directly from the inequality $V(fg) \le V(f) + V(g)$ for f and g such that $|f| \equiv |g| \equiv 1$. □

LEMMA 4. — *There exists a set I_β of irrational α of full measure such that for α in I_β we can find a constant C_α and a number s, $0 < s < 1$, such that*

$$\left|\int_0^1 f_h(t)dt\right| = \left|\int_0^1 e^{2\pi i q\beta(\{t\}+\{t+\alpha\}+\cdots+\{t+(h-1)\alpha\})}dt\right| \le \frac{C_\alpha}{h^s}.$$