Bull. Soc. math. France
129 (3), 2001, p. 449–485

SINGULARITIES OF 2Θ-DIVISORS IN THE JACOBIAN

BY CHRISTIAN PAULY & EMMA PREVIATO

ABSTRACT. — We consider the linear system $|2\Theta_0|$ of second order theta functions over the Jacobian JC of a non-hyperelliptic curve C. A result by J. Fay says that a divisor $D \in |2\Theta_0|$ contains the origin $\mathcal{O} \in JC$ with multiplicity 4 if and only if D contains the surface $C - C = \{\mathcal{O}(p-q) \mid p, q \in C\} \subset JC$. In this paper we generalize Fay's result and some previous work by R.C. Gunning. More precisely, we describe the relationship between divisors containing $\mathcal{O}$ with multiplicity 6, divisors containing the fourfold $C_2 - C_2 = \{\mathcal{O}(p+q-r-s) \mid p, q, r, s \in C\}$, and divisors singular along $C - C$, using the third exterior product of the canonical space and the space of quadrics containing the canonical curve. Moreover we show that some of these spaces are equal to the linear span of Brill-Noether loci in the moduli space of semi-stable rank 2 vector bundles with canonical determinant over C, which can be embedded in $|2\Theta_0|$.

Texte reçu le 9 mars 2000, révisé le 28 novembre 2000

CHRISTIAN PAULY, Laboratoire J.-A. Dieudonné, Université de Nice Sophia Antipolis, Parc Valrose, F-06108 Nice Cedex 02, France • *E-mail :* `pauly@math.unice.fr`
EMMA PREVIATO, Department of Mathematics, Boston University, Boston, MA 02215-2411, U.S.A. • *E-mail :* `ep@math.bu.edu`

2000 Mathematics Subject Classification. — 14H42,14H40, 14H60.

Key words and phrases. — Theta functions, Jacobian, canonical curve, vector bundle.

Research partly supported by NSF grant DMS 9971966.

RÉSUMÉ (*Singularités des diviseurs* 2Θ *d'une jacobienne*). — On considère le système linéaire $|2\Theta_0|$ des fonctions thêta d'ordre deux sur la jacobienne JC d'une courbe non-hyperelliptique C. Un résultat de J. Fay affirme qu'un diviseur $D \in |2\Theta_0|$ contient l'origine $\mathcal{O} \in JC$ avec multiplicié 4 si et seulement si D contient la surface $C - C = \{\mathcal{O}(p-q) \mid p,q \in C\} \subset JC$. Dans cet article on généralise le résultat de Fay ainsi que quelques travaux de R.C. Gunning. On décrit la relation entre les diviseurs contenant $\mathcal{O}$ avec multiplicité 6, les diviseurs contenant la sous-variété $C_2 - C_2 = \{\mathcal{O}(p+q-r-s) \mid p,q,r,s \in C\}$, et les diviseurs singuliers le long de $C - C$, en utilisant la troisième puissance extérieure de l'espace canonique et l'espace des quadriques contenant la courbe canonique. De plus on montre que certains sous-systèmes linéaires sont isomorphes aux enveloppes linéaires de lieux de Brill-Noether dans l'espace de modules des fibrés vectoriels semi-stables de rang 2 et de déterminant canonique, qui sont plongés dans $|2\Theta_0|$.

1. Introduction

Let C be a smooth, connected, projective non-hyperelliptic curve of genus $g \geq 3$ over the complex numbers and let $\mathrm{Pic}^d(C)$ be the connected component of its Picard variety parametrizing degree d line bundles, for $d \in \mathbb{Z}$. The variety $\mathrm{Pic}^{g-1}(C)$ carries a naturally defined divisor, the Riemann theta divisor Θ, whose support consists of line bundles that have nonzero global sections. Translating Θ by a theta characteristic, we obtain a symmetric theta divisor, denoted Θ_0, on the Jacobian variety of C,

$$JC := \mathrm{Pic}^0(C).$$

Our principal objects of study are the linear systems of 2θ-divisors $|2\Theta_0|$ over JC and $|2\Theta|$ over $\mathrm{Pic}^{g-1}(C)$, their linear subspaces and subvarieties. One of the features of these linear systems is the canonical duality, called Wirtinger duality (see [21], p. 335), which we will use somewhat implicitly throughout this paper

$$w : |2\Theta|^* \cong |2\Theta_0|. \tag{1.1}$$

Because of (1.1) we can view the Kummer variety $\mathrm{Kum} := JC/\pm$ as a subvariety of $|2\Theta|$ and many classical aspects of its projective geometry, such as existence of trisecants, tangent cones at its singular points [7], [13], can be expressed in terms of 2θ-functions. The starting point of our investigations of 2θ-divisors is the following remarkable equivalence which was observed by J. Fay (see *e.g.* [26], Prop. 4.8 and [13], Cor. 1)

$$\mathrm{mult}_0(D) \geq 4 \iff C - C \subset D, \qquad \forall D \in |2\Theta_0|, \tag{1.2}$$

where the surface $C - C$ denotes the image of the difference map

$$\phi_1 : C \times C \longrightarrow JC,$$

which sends a pair of points (p,q) to the line bundle $\mathcal{O}(p-q)$. Motivated by (1.2) van Geemen and van der Geer [8] introduce the subseries $\mathbb{P}\Gamma_{00} \subset$

$|2\Theta_0|$ consisting of 2θ-divisors having multiplicity at least 4 at the origin and formulate a number of Schottky-type conjectures, some of which have been proved (see [26] and [15]).

We can reformulate (1.2) more geometrically. Let

$$\langle C - C\rangle \subset |2\Theta|$$

be the linear span of the image of $C - C$ in $|2\Theta|$ and let $\mathbb{T}_0 \subset |2\Theta|$ be the embedded tangent space at the singular point $\mathcal{O}$ to the Kummer variety. Then (see [9], Lemma 1.5 and [26], Prop. 4.8) these linear projective spaces coincide

$$\mathbb{T}_0 = \langle C - C\rangle.$$

Note that their polar space in $|2\Theta|^*$ ($\cong |2\Theta_0|$) is $\mathbb{P}\Gamma_{00}$. More precisely, if we denote by $\Gamma_0 \subset H^0(JC, \mathcal{O}(2\Theta_0))$ the hyperplane of 2θ-divisors containing $\mathcal{O}$, we obtain an isomorphism by restricting divisors to $C - C$

$$\Gamma_0/\Gamma_{00} \xrightarrow{\sim} \mathrm{Sym}^2 H^0(K). \tag{1.3}$$

Having the equivalence (1.2) in mind, we can ask whether there exist relations between higher order derivatives of 2θ-functions at the origin $\mathcal{O}$ and natural subvarieties of JC. Working in an analytic set-up, Gunning (see [13], Section 8, [12], Section 9) establishes some linear relations between vectors of 2θ-functions. Inspired by Gunning's previous work, we will compare the following subseries of $\mathbb{P}\Gamma_{00}$, using algebraic methods

$$\mathbb{P}\Gamma_{11} = \big\{D \in \mathbb{P}\Gamma_{00} \mid C_2 - C_2 \subset D\big\}, \tag{1.4}$$

$$\mathbb{P}\Gamma_{000} = \big\{D \in \mathbb{P}\Gamma_{00} \mid \mathrm{mult}_0(D) \geq 6\big\}. \tag{1.5}$$

The fourfold $C_2 - C_2$ is defined to be the image of the difference map $\phi_2 : C_2 \times C_2 \to JC$, which maps a 4-tuple $(p+q, r+s)$ to the line bundle $\mathcal{O}(p+q-r-s)$ and C_2 is the second symmetric product of the curve. Moreover we will be naturally led to consider the subseries of 2θ-divisors which are singular along the surface $C - C$, *i.e.*,

$$\mathbb{P}\Gamma_{00}^{(2)} = \big\{D \in \mathbb{P}\Gamma_{00} \mid \mathrm{mult}_{p-q}(D) \geq 2 \ \forall p, q \in C\big\}. \tag{1.6}$$

We observe that the subseries $\mathbb{P}\Gamma_{00}, \mathbb{P}\Gamma_{11}, \mathbb{P}\Gamma_{00}^{(2)}$ are closely related to the geometry of the moduli space $\mathcal{SU}_C(2, K)$ of semi-stable rank 2 vector bundles with fixed canonical determinant. The morphism (Section 2)

$$D : \mathcal{SU}_C(2, K) \longrightarrow |2\Theta_0|, \qquad E \longmapsto D(E) = \big\{\xi \in JC \mid h^0(E \otimes \xi) > 0\big\}$$

was recently shown [6], [9] to be an embedding. Hence we may view $\mathcal{SU}_C(2, K)$ as a subvariety of $|2\Theta_0|$. Of considerable interest are the Brill-Noether loci $\mathcal{W}(n) \subset \mathcal{SU}_C(2, K)$ for $n \geq 1$ defined by

$$\mathcal{W}(n) = \big\{[E] \in \mathcal{SU}_C(2, K) \mid h^0(E) = n \text{ and } E \text{ is globally generated}\big\}.$$

These loci (more precisely their closure) have been extensively studied in connection with Fano threefolds [20], [19], [18] and for their own sake [22]. A simple argument now shows that one has the following implications

$$[E] \in \mathcal{W}(2) \quad (\text{resp. } \mathcal{W}(3), \mathcal{W}(4), \mathcal{W}(5))$$
$$\Longrightarrow \quad D(E) \in \mathbb{P}\Gamma_0 \quad (\text{resp. } \mathbb{P}\Gamma_{00}, \mathbb{P}\Gamma_{00}^{(2)}, \mathbb{P}\Gamma_{11})$$

We see that we obtain two filtrations which are related under the map D, one given by the dimension of the space of sections of $E \in \mathcal{S}U_C(2,K)$, the other given by 2θ-divisors containing certain subschemes of JC. As a consequence of our results we see that the Brill-Noether loci $\mathcal{W}(n)$ for $n = 2,3,4$ linearly span the corresponding subseries. We expect this to hold for $n = 5$ too.

In the next two theorems we describe the first two quotients of the filtration

$$\Gamma_{11} \subset \Gamma_{00}^{(2)} \subset \Gamma_{00} \subset \Gamma_0,$$

the last one being given in (1.3). Let $\langle \mathrm{Sing}\Theta \rangle$ be the linear span of the image of the singular locus $\mathrm{Sing}\,\Theta \subset \mathrm{Pic}^{g-1}(C)$ under the morphism into $|2\Theta_0|$.

THEOREM 1.1. — *For any non-hyperelliptic curve*

1) *there exists a canonical isomorphism (up to a scalar)*

$$\Gamma_{00}/\Gamma_{00}^{(2)} \xrightarrow{\sim} \Lambda^3 H^0(K).$$

2) *we have an equality among subspaces of* $\mathbb{P}\Gamma_{00}$

$$\langle \mathrm{Sing}\Theta \rangle = \mathbb{P}\Gamma_{00}^{(2)}.$$

The method used in the proof of this theorem (Section 4) has been developed in a recent paper by van Geemen and Izadi [9] and the key point is the incidence relations (Section 2.4) between two families of stable rank 2 vector bundles with fixed trivial (resp. canonical) determinant. One of these families of bundles is related to the gradient of the 2θ functions along the surface $C - C$, the other family is the Brill-Noether locus $\mathcal{W}(3)$. In Section 2.5 we describe the relationship between these bundles and the objects discussed in [9], which are related to the embedded tangent space at the origin to $\mathcal{S}U_C(2,K)$. We also need (Section 3) some relations between vectors of second order theta functions, which one derives from Fay's trisecant formula and its generalizations [13].

Let $I(2)$ (resp. $I(4)$) be the space of quadrics (resp. quartics) in canonical space $|K|^*$ containing the canonical curve.

THEOREM 1.2. — *For any non-trigonal curve, there exists a canonical isomorphism*

$$\Gamma_{00}^{(2)}/\Gamma_{11} \xrightarrow{\sim} \mathrm{Sym}^2 I(2). \tag{1.7}$$

The harder statement in Theorem 1.2 is the surjectivity of the map in (1.7). The proof uses essentially two ideas: first, we can give an explicit basis of quadrics in $I(2)$ of rank less than or equal to 6 (Petri's quadrics, Section 5.1) and secondly, we can construct out of such a quadric a rank 2 vector bundle in $\mathcal{W}(4)$. This construction [6] is recalled in Section 5.2 and generalized in Section 8. As a corollary of Theorem 1.2 (Section 5.4), we obtain another proof of a theorem by M. Green, saying that the projectivized tangent cones to Θ at double its points span $I(2)$.

The subspace Γ_{000} is of a different nature and rank 2 vector bundles turn out to be of no help in studying it. We gather our results in the next theorem.

THEOREM 1.3. — *For any non-hyperelliptic curve, we have the following inclusions*

$$\Gamma_{11} \subset \Gamma_{000} \subset \Gamma_{00}^{(2)}. \tag{1.8}$$

The quotient of the first two spaces is isomorphic to the kernel of the multiplication map m

$$\Gamma_{000}/\Gamma_{11} \cong \ker m : \mathrm{Sym}^2 I(2) \longrightarrow I(4). \tag{1.9}$$

The proof of Theorem 1.3 is more in the spirit of Gunning's previous work and uses only linear relations between vectors of second order theta functions. The inclusions (1.8) were proposed as plausible in [13], p. 70. Except for a few cases (Section 6.2) we are unable to deduce the dimension of Γ_{000} from (1.9).

In Section 7 we give the version of Theorem 1.2 for trigonal curves.

We observe that the vector bundle constructions used in the proofs of Theorems 1.1 and 1.2 can be seen as examples of a global construction (Section 8) which relates a bundle in $\mathcal{W}(n)$ to the geometry of the canonical curve.

Acknowledgements. — The authors are grateful to K. Chandler, J. Fay, W.M. Oxbury and A. Verra for various helpful comments. We especially thank B. van Geemen for many fruitful discussions and his constant interest in this work.

Notation

– If X is a vector space or a vector bundle, by X^* we denote its dual.

– K is the canonical line bundle on the curve C.

– For a vector bundle E over C, $H^i(C,E)$ is often abbreviated by $H^i(E)$ and $h^i(E) = \dim H^i(C,E)$.

– C_n is the n-th symmetric product of the curve C.

– $W_d^r(C)$ is the subvariety of $\mathrm{Pic}^d(C)$ consisting of line bundles L such that $h^0(L) > r$.

– The canonical curve C_{can} is the image of the embedding $\varphi_K : C \longrightarrow |K|^*$.

– The vector space $I(n)$ is the space of forms in $\mathrm{Sym}^n H^0(K)$ defining degree n hypersurfaces in $|K|^*$ containing C_{can}.