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POLYNOMIAL DECAY OF CORRELATIONS FOR A CLASS OF SMOOTH FLOWS ON THE TWO TORUS

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Abstract. — Kočergin introduced in 1975 a class of smooth flows on the two torus that are mixing. When these flows have one fixed point, they can be viewed as special flows over an irrational rotation of the circle, with a ceiling function having a power-like singularity. Under a Diophantine condition on the rotation's angle, we prove that the special flows actually have a $t^{-\eta}$-speed of mixing, for some $\eta > 0$.

Résumé (*Décroissance polynomiale des corrélations pour une classe de flots lisses sur* $\mathbb{T}^2$)

Kočergin a introduit en 1975 une classe de flots C^∞ sur le tore à deux dimensions qui sont mélangeants. Quand ces flots ont un seul point fixe, ils correspondent à des flots spéciaux au-dessus d'une rotation irrationnelle du cercle, dont la fonction de suspension présente une singularité en puissance fractionnaire. Sous une condition diophantienne sur l'angle de la rotation, on prouve que ces flots spéciaux ont une vitesse de mélange en $t^{-\eta}$, pour un certain $\eta > 0$.

1. Introduction

1.1. Kočergin gave in [5], examples of C^∞ measure-preserving flows on the two torus that are mixing. He starts by proving that special flows over irrational rotations of the circle (or over interval exchange transformations) with

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a ceiling function having a power-like singularity are mixing. It is possible to identify these special flows with smooth flows on the two torus, having one fixed point or more. In the same article, Kočergin describes in some examples how to realize this identification by smoothly gluing to an irrational flow $R_{t(1,\alpha)}$ a small neighborhood of the fixed point of an adequatly chosen Hamiltonian flow in the plane. We will prove that under a Diophantine condition on the rotation number, the special flow has a polynomial decay of correlations between rectangles. We will see later why the arithmetical condition is required.

1.2. First we give the definition of a special flow over an irrational rotation. Given a strictly positive function $\varphi \in L^1(\mathbb{T}^1)$, *the special flow constructed over* R_α *and under the function* φ is the quotient flow of the action

$$\begin{aligned} \mathbb{T}^1 \times \mathbb{R} &\longrightarrow \mathbb{T}^1 \times \mathbb{R}, \\ (x,s) &\longrightarrow (x, s+t), \end{aligned}$$

by the relation

$$\big(x, s+\varphi(x)\big) \sim \big(R_\alpha(x), s\big).$$

This flow acts on the space

$$M_{R_\alpha,\varphi} = \mathbb{T}^1 \times \mathbb{R}/\sim,$$

is uniquely ergodic and preserves the normalized Lebesgue measure on $M_{R_\alpha,\varphi}$, *i.e.* the product of the Haar measure on the basis $\mathbb{T}^1$ with the Lebesgue measure on the fibers divided by the constant $\int_{\mathbb{T}^1} \varphi(x)\,\mathrm{d}x$. In the sequel, we will simply denote by M the space $M_{R_\alpha,\varphi}$, and by μ the invariant measure described above.

We call *rectangles* in M the sets

$$B = \bigcup_{t=t_0}^{t_0+\ell} T^t(I),$$

when the union is disjoint and I is an interval of $\mathbb{T}^1$, $t_0 \in \mathbb{R}$ and $\ell \in \mathbb{R}_+^*$.

It is immediate that the collection of rectangles generates the σ-algebra of Borel sets on M. There is of course a slight abuse in calling rectangles the latter sets, because under the action of the flow, when t_0 or ℓ are large, they get distorted and do not have rectangular shapes anymore. Nevertheless, when $t_0 = 0$ and $\ell \le \inf_{\mathbb{T}^1} \varphi$, we have real rectangles that we call *rectangles on the basis.*

1.3. Description of the flow under consideration. — In what follows we will consider the special flow $\{T^t\}$ constructed over an irrational rotation R_α, and under a ceiling function φ.

We assume the following hypothesis on φ:

- $\varphi \in C^3(\mathbb{T}/\{0\})$, and $\inf_{\mathbb{T}^1} \varphi = c > 0$;
- there exists $0 < \gamma < 1$ such that

$$\begin{aligned}
&\lim_{x\to 0^+} \frac{\varphi(x)}{x^{-\gamma}} = 1, && \lim_{x\to 0^-} \frac{\varphi(x)}{(-x)^{-\gamma}} = 1,\\
&\lim_{x\to 0^+} \frac{\varphi'(x)}{-\gamma x^{-\gamma-1}} = 1, && \lim_{x\to 0^-} \frac{\varphi'(x)}{\gamma(-x)^{-\gamma-1}} = 1,\\
&\lim_{x\to 0^+} \frac{\varphi''(x)}{\gamma(\gamma+1)x^{-\gamma-2}} = 1, && \lim_{x\to 0^-} \frac{\varphi''(x)}{\gamma(\gamma+1)(-x)^{-\gamma-2}} = 1.
\end{aligned}$$

For commodity, we will also suppose that $\int_{\mathbb{T}^1} \varphi(x)\,\mathrm{d}x = 1$.

It follows from Kočergin's result that these special flows, for any α and any $\gamma \in\,]0,1[$, are mixing. But to force estimates on the decay of correlations, our techniques require that the exponent should be at least less than $\frac{1}{2}$, and we will assume $\gamma \leq \frac{2}{5}$.

Remark. — The exponent obtained in one of the smooth examples given by Kočergin is $\gamma = \frac{1}{3} < \frac{2}{5}$.

As for the rotation number α, we require that the sequence $\{q_n\}_{n\in\mathbb{N}}$ of denominators of its convergents satisfies for some positive constant C_α

$$q_{n+1} \leq C_\alpha {q_n}^{1+\epsilon}, \tag{CD-ϵ}$$

where ϵ is small compared to γ, $\epsilon = \frac{1}{100}\gamma$ being enough for our purpose.

Finally, we introduce the number

$$\eta := \frac{1}{50}\gamma.$$

In the sequel, we will often use the fact that $2\epsilon \leq \eta \ll \gamma$.

1.4. Statement. — Under the above assumptions on α and φ, we will show the following:

Theorem 1.1. — *For any two rectangles A and B, for any $\eta_0 < \eta$, we have for t large enough*

$$\bigl|\mu(A \cap T^{-t}B) - \mu(A)\mu(B)\bigr| \leq \frac{1}{t^{\eta_0}}\cdot \tag{1}$$

In the statement, the rectangles and the measure μ are as defined in (1.2).

1.5. Remarks. — For a fixed $\epsilon > 0$, the set of rotation numbers satisfying a Diophantine condition (CD-ϵ) is of total Lebesgue measure.

To simplify the presentation we considered only one power-like and symmetrical singularity

$$\varphi(x) \underset{0}{\sim} \varphi(-x) \underset{0}{\sim} |x|^{-\gamma}.$$

From the proof it will appear clearly that the same result holds when there are finitely many singularities, and not all of them being necessarily power-like (some of them could be logarithmic for example), under the condition that the strongest one should be power-like (with exponent $\gamma \leq \frac{2}{5}$). Furthermore, our assumption of symmetry is not necessary, to the contrary, symmetry plays in general against mixing. When the singularity is logarithmic for example, the symmetry impedes mixing as proved by Kočergin in [4]; while Khanin and Sinai proved mixing in the case of asymmetrical logarithmic singularities [3].

Our estimates are far from optimal and $\eta = \frac{\gamma}{50}$ is certainly not the best polynomial rate of decay one can obtain. A faster speed of mixing than $t^{-\frac{1}{2}-\epsilon}$ would be very interesting because it would imply a Lebesgue spectrum for the flow. But since it appeared very hard to obtain faster decay than $t^{-\frac{1}{2}}$ by the techniques involved in this paper we wrote the proof with the above $\eta = \frac{\gamma}{50}$.

Correlations between functions. — Through the proof of the theorem, it will appear that (1) is valid when $t \in [q_n, q_{n+1}]$, n sufficiently large, for any pair of squares A and B with side of length equal to $q_n^{-\eta}$. Hence we could establish for any couple of complex functions of class C^1 on $\mathbb{T}^2$, the same decay of correlations obtained for rectangles.

1.6. Plan of the proof. — The property underlying mixing for a special flow over a rigid transformation is the *uniform stretch* of the Birkhoff sums of the ceiling function,

$$\varphi_m(x) := \varphi(x) + \varphi\big(R_\alpha(x)\big) + \cdots + \varphi\big(R_\alpha^{m-1}(x)\big).$$

When the Birkhoff sums have large derivatives, the image of a small interval $J \in \mathbb{T}^1$ by the flow is stretched with time in the vertical direction along the fibers, and as t tends to infinity the interval actually breaks down into a lot of almost vertical curves whose projections on the circle follow the trajectory of R_α. By unique ergodicity of the rotation these projections become more uniformly distributed on the circle as their number increases, and so will be $T^t(J)$ in the whole space (see [6], [5], [3], [2] and [1]).

For each t, we want to cover the circle excepted a small set with intervals being stretched as described above (here, we want the exceptional set to have measure less than $t^{-\eta}$). The first intervals to be over ruled are those that come too close to the singularity before time t and eventually get trapped in its neighborhood (Lemma 2.3). Other intervals must be automatically discarded, those where there is no stretch at all, *i.e.* where the first derivative of φ_m is small (the singularity being symmetric this likely occurs): a lower bound on the second derivatives due to the convex average behavior of φ will allow us to estimate the size of such bad intervals (Lemma 2.5 and Step 2 in Section 3.1).

For the remaining part of the circle, we seek a good control on the stretch of φ_m, when m is comparable with t, and here the Diophantine condition on α is required to insure the *uniformity* of stretch (Lemmas 2.4 and 2.5). Still, uniform stretch (Properties (P1)–(P2′) in (3.1)) of an interval J is not enough by itself to estimate the asymptotic repartition of $T^t(J)$ in the space as t goes to infinity. We need in addition to make sure that the pieces of $T^t(J)$ (those almost vertical curves) do not enter in a too small neighborhood of the singularity, otherwise a lot of measure can be lost there (See (2.4)).

A "good" partition is finally constructed for each time t (Proposition 3.1) and Lemmas 3.3 and 3.2 give a precise description of $T^t(J)$, for an interval J in this partition.

To conclude, we need a good estimate on the asymptotic distribution of the trajectories of the rotation on the circle that we obtain using again the Diophantine condition on α.

2. Preliminary estimates and lemmas

2.1. A Fubini Lemma. — We begin by a Fubini lemma that reduces our problem to studying the image under the special flow of intervals on its basis $\mathbb{T}^1$.

Given $\nu > 0$ and a finite partial partition $\mathcal{P} = \{I_0, ..., I_m\}$ of $\mathbb{T}^1$, we say that $\mathcal{P}$ is ν-*fine* if, for any interval I on the circle, there exists a collection of atoms from $\mathcal{P}$ such that the symmetrical difference between their union and I has Lebesgue measure less than ν.

We recall that a rectangle on the basis is a subset $B = \bigcup_{0\le t\le \ell} T^t(I)$, where I is an interval of $\mathbb{T}^1$ and $\ell \in \mathbb{R}_+^*$ is the height of B. We also recall that η is the fixed number $\frac{1}{50}\gamma$ $(2\epsilon \le \eta \ll \gamma \le \frac{2}{5})$.

Lemma 2.1. — *If there exist partial partitions of $\mathbb{T}^1$, $\mathcal{P}_t$, that are $t^{-\eta}$-fine and such that for any rectangle B on the basis with height less than $c = \inf_{\mathbb{T}^1}\varphi$, we have when t is large enough*

$$\big|\lambda(J_i^{(t)} \cap T^{-t}B) - \lambda(J_i^{(t)})\mu(B)\big| \le t^{-\eta}\lambda(J_i^{(t)}), \tag{2}$$

for every $J_i^{(t)} \in \mathcal{P}_t$, then Theorem 1.1 *is true.*

In the statement of the Lemma, λ is the Haar measure on the circle and μ is the normalized measure on M invariant by the special flow.