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A MEAN-VALUE LEMMA AND APPLICATIONS

by Alessandro Savo

Abstract. — We control the gap between the mean value of a function on a submanifold (or a point), and its mean value on any tube around the submanifold (in fact, we give the exact value of the second derivative of the gap). We apply this formula to obtain comparison theorems between eigenvalues of the Laplace-Beltrami operator, and then to compute the first three terms of the asymptotic time-expansion of a heat diffusion process on convex polyhedrons in euclidean spaces of arbitrary dimension. We also write explicit bounds for the remainder term of the above expansion, which hold for all values of time. The results of this paper have been announced, without proof, in [16].

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Résumé (*Un lemme de valeur moyenne et quelques applications*)
On contrôle l'écart entre la valeur moyenne d'une fonction sur une sous-variété d'une variété riemannienne, et sa valeur moyenne sur un voisinage tubulaire autour de la sous-variété (on donne, en effet, la valeur exacte de la dérivée seconde de cet écart). On applique ensuite cette formule afin d'obtenir des théorèmes de comparaison pour les valeurs propres et les fonctions propres de l'opérateur de Laplace-Beltrami, et pour calculer les trois premiers termes du développement asymptotique relatif à un problème de diffusion de la chaleur sur les polyèdres convexes dans un espace euclidien de dimension quelconque. On donne enfin des bornes explicites des restes du développement susdit, qui sont valable pour toute valeur du temps. Les résultats de cet article ont été annoncés (sans démonstrations) dans [16].

Contents

1. Introduction

Section 2 contains the technical background of the paper. Let N be a compact, piecewise-smooth submanifold of the complete, n-dimensional Riemannian manifold M. The *tube* of radius r around N is the set

$$M(r) = \{x \in M : \rho(x) < r\},$$

where ρ is the *distance function from* N. Given a function u on M, our aim is to describe, in Theorem 2.5, the second derivative of the function

$$F(r) = \int_{M(r)} u \, \mathrm{d}v_n$$

where $r > 0$, and where $\mathrm{d}v_n$ is the volume form on M given by the metric. This is equivalent to estimate

$$\frac{F(r)}{\mathrm{vol}(M(r))} - \int_N u,$$

and thus it may be seen as a generalization of the classical mean-value lemma, which says that, when M is the euclidean space and $N = \{x_0\}$, any harmonic function satisfies $F(r)/\mathrm{vol}(M(r)) = u(x_0)$ for all r.

For an arbitrary function u, it turns out that the second derivative of F involves the Laplacian of u, as well as the Laplacian of the distance function ρ. Now, if we stay within the injectivity radius of N, *i.e.* if we stay away from the cut-locus of N in M, both ρ and F will be smooth functions (of $x \in M$ and r respectively); however, the nature of the problems we intend to investigate (which include the piecewise-smooth case) forced us to take into account also the points of the cut-locus, and then to consider $F(r)$ as a singular function on the whole half-line.

Due to the cut-locus, both F and ρ are only Lipschitz regular, and their Laplacians must therefore be taken in the sense of distributions. Hence, our first preoccupation will be to describe, in Lemma 2.1, the distributional Laplacian of the distance function, and to show that it decomposes in a regular part $\Delta_{\mathrm{reg}}\rho$ (an L^1_{loc}-function on M), and a singular part, which is in turn the sum of a positive Radon measure $\Delta_{\mathrm{Cut}}\rho$, supported on the cut-locus of N, and the Dirac measure $-2\delta_N$, supported on the submanifold N and vanishing when N has codimension greater than 1.

As a preparatory step, we prove a version of Green's theorem for the (generally singular) tubes $M(r)$ (Proposition 2.3); and we then proceed with the proof of the main technical lemma, called the *Mean-value Lemma* (see Theorem 2.5):

$$-F''(r) = \int_{M(r)} \Delta u \,\mathrm{d}v_n + \rho_*(u\Delta\rho)(r), \tag{1}$$

where ρ_* is the operator of push-forward on distributions, which is dual to the pull-back operator ρ^*. (If $r = \rho(x)$ is smaller than the injectivity radius of N, then $\Delta\rho$ is smooth at x, and gives the trace of the second fundamental form of the hypersurface $\rho^{-1}(r)$ at x; in that case, $\rho_*(u\Delta\rho)(r) = \int_{\rho^{-1}(r)} u\Delta\rho$, the integration being performed with respect to the induced measure on $\rho^{-1}(r)$).

Section 3 deals with the applications of Theorem 2.5 to *eigenvalue estimates*. Some of the results exposed here are already known, but the proofs we provide are, we believe, new, and we have chosen to include them to show the usefulness of our approach, which gives a simple unified proof of all these results. So let us select an eigenfunction u of the Laplace-Beltrami operator,

$$\Delta u = \lambda u,$$

and let

$$F(r) = \int_{M(r)} u.$$

Theorem 2.5 becomes the following statement:

$$-F'' = \lambda F + \rho_*(u\Delta\rho). \tag{2}$$

If u is harmonic, and if all the geodesic spheres of M around x_0 have constant mean curvature (in particular, if M is a manifold of revolution around x_0, or if M is a symmetric space) then one can immediately re-derive the "classical mean-value lemma" by applying (2) in the case where ρ is the distance from x_0.

The basic idea in the use of equation (2) is that it is possible to bound from below the distribution $\Delta\rho$ by an explicit radial function on M (that is, a function which depends only on the distance from N), if one assumes in addition a lower bound of the Ricci curvature on M. Then we derive from (2) a second order differential inequality in F, which can be studied by standard comparison arguments. We explicitly carry out the idea in the following two cases: when ρ is the distance from a point, and when ρ is the distance from the boundary of a domain.

Let us apply principle (2) when $N = \{x_0\}$. Assume that Ricci $\geq (n-1)K$, where K is any real number. Let $B(x_0, R)$ (resp. $\overline{B}(R)$) be a geodesic ball of radius R in M (resp. in the simply connected manifold $\overline{M}_K$ of constant curvature K). We then obtain, in Theorem 3.1, for any positive solution of

$$\Delta u \geq \lambda u \quad \text{on} \quad B(x_0, R)$$

(resp. for any positive solution of $\Delta\bar{u} = \lambda\bar{u}$ on $\overline{B}(R)$), the following inequality

$$\frac{\int_{\partial B(x_0,r)} u}{\int_{B(x_0,r)} u} \leq \frac{\int_{\partial\overline{B}(r)} \bar{u}}{\int_{\overline{B}(r)} \bar{u}}$$

for all $0 < r < R$. Theorem 3.1 reduces to the classical Bishop-Gromov inequality if $u = \bar{u} = 1$. Notice that R is not assumed to be smaller than the injectivity radius of x_0, so that the above inequality extends *beyond* the cut-locus of x_0.

We observe two consequences of Theorem 3.1: the first (Corollary 3.3), states that if u is a positive superharmonic function on $B(x_0, R)$, then, for $0 < r < R$, we have

$$u(x_0) \geq \frac{1}{\operatorname{vol}\partial\overline{B}(r)} \int_{\partial B(x_0,R)} u,$$

and the second (Theorem 3.4) is a well-known inequality of Cheng's regarding the first eigenvalues of the Dirichlet Laplacian on open balls in M and $\overline{M}$ respectively: $\lambda_1(B(R)) \leq \lambda_1(\overline{B}(R))$ which is proved in [6], by different methods.

In the second part of Section 3, we use equation (1) in the case where ρ is the distance function from the boundary of a domain Ω in M. We assume a lower bound $\bar{\eta}$ for the mean curvature of $\partial\Omega$, a lower bound $(n-1)K$ for the Ricci curvature of $\partial\Omega$, and we denote by R the *inner radius* of Ω (that is, the radius of the biggest ball that fits into Ω). We then consider the "symmetrized" domain $\overline{\Omega}$ corresponding to the data $\bar{\eta}, K, R$: it will be the cylinder of constant curvature K, and width R, having constant mean curvature equal to $\bar{\eta}$ on one, say Γ, of the two connected components of the boundary. We then show, in Theorem 3.6, that

$$\lambda_1(\Omega) \geq \lambda_1(\overline{\Omega})$$

where $\lambda_1(\Omega)$ is the first eigenvalue of the Dirichlet problem on Ω, and $\lambda_1(\overline{\Omega})$ is the first eigenvalue of the following mixed problem on $\overline{\Omega}$: Dirichlet condition on the component Γ, Neumann condition on the other. The result extends to any domain with piecewise-smooth boundary satisfying an additional property (see Property (P), before Lemma 3.5), and should be compared with the corresponding result obtained by Kasue [13], by different methods. In the special case $\bar{\eta} = 0$, $K = 0$, Theorem 3.6 reduces to the well-known inequality $\lambda_1(\Omega) \geq \pi^2/4R^2$, due to Li and Yau (see [15], Theorem 11).

Section 4 deals with the applications of the Mean-Value Lemma to *heat diffusion.* We fix a domain Ω (we assume $\partial\Omega$ piecewise-smooth), and we fix the solution $u(t,x)$ of the heat equation on Ω satisfying Dirichlet boundary conditions, and having unit initial conditions ($u(0,x) = 1$ for all $x \in \Omega$). We call $u(t,x)$ the *temperature* function of Ω. Integrating it over Ω, we obtain the *heat content* function $H(t)$:

$$H(t) = \int_\Omega u(t,x)\,\mathrm{d}x.$$

The function $H(t)$ has been the object of investigation by a number of authors (see [1], [2], [3]); its importance lies also in the fact that, if one denotes by $k(t,x,y)$ the heat kernel of the domain Ω relative to Dirichlet boundary conditions, $H(t)$ is the integral on $\Omega \times \Omega$ of $k(t,\cdot\,,\cdot)$ with respect to the product measure.

Our basic idea in dealing with $H(t)$ is to introduce an auxiliary variable $r \geq 0$, and then consider the map

$$H(t,r) = \int_{\Omega(r)} u(t,x)\,\mathrm{d}x,$$

where

$$\Omega(r) = \bigl\{x \in \Omega : d(x,\partial\Omega) > r\bigr\}$$

are the *parallel domains* of Ω. By the Mean-value Lemma, applied for $N = \partial\Omega$, we immediately obtain that $H(t,r)$ satisfies a heat equation on the half-line $(0,\infty)$, of the type

$$\Bigl(-\frac{\partial^2}{\partial r^2} + \frac{\partial}{\partial t}\Bigr)H = -\rho_*\bigl(u(t,\cdot)\Delta\rho\bigr).$$

The main advantage of the method is that it reduces the problem to a *one-dimensional* one, where all computations can be performed explicitly: in fact, using Duhamel principle (Lemma 4.1), we can represent the heat content $H(t)$ in terms of the measure $\rho_*((1-u(t,\cdot))\Delta\rho)$ and in terms of the Neumann