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PROOF OF NADEL'S CONJECTURE AND DIRECT IMAGE FOR RELATIVE K-THEORY

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ABSTRACT. — A "relative" K-theory group for holomorphic or algebraic vector bundles on a compact or quasiprojective complex manifold is constructed, and Chern-Simons type characteristic classes are defined on this group in the spirit of Nadel. In the projective case, their coincidence with the Abel-Jacobi image of the Chern classes of the bundles is proved. Some applications to families of holomorphic bundles are given and two Riemann-Roch type theorems are proved for these classes.

RÉSUMÉ (*Démonstration d'une conjecture de Nadel et image directe pour la K-théorie relative*)

On construit un groupe de K-théorie relative pour les fibrés holomorphes ou algébriques sur une variété complexe compacte ou quasiprojective, et des classes caractéristiques de type de Chern-Simons sont définies sur ce groupe dans l'esprit de Nadel. Dans le cas projectif, on démontre la coïncidence de ces classes avec l'image par l'application d'Abel-Jacobi des classes de Chern des fibrés. On donne quelques applications aux familles de fibrés holomorphes et on démontre deux théorèmes de type Riemann-Roch pour ces classes.

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1. Introduction

On a compact complex manifold X, Nadel defined in [35] characteristic classes à la Chern-Weil with values in the odd Dolbeault cohomology of X for triples (E,F,f) where E and F are holomorphic vector bundles and $f\colon E \xrightarrow{\sim} F$ is a $\mathcal{C}^\infty$ isomorphism. He then conjectured that his classes should coïncide with some part of the Abel-Jacobi image of the cycle valued Chern character (without denominators) of the difference $[F]-[E]\in K_0(X)$ (here $K_0(X)$ is the Grothendieck group of algebraic vector bundles modulo exact sequences).

In this paper, Nadel's theory is widely generalised. Firstly, some relative K-theory group $K_0^{\mathrm{rel}}(X)$ is constructed (in Section 2) whose elements are equivalence classes of objects (E,F,f) as above; the construction is also valid for couples of $\mathcal{C}^\infty$ isomorphic algebraic vector bundles on quasiprojective manifolds. Let $K_1(X)$ be Quillen's algebraic K-theory group of the category of holomorphic (algebraic) vector bundles on X, $K_0^{\mathrm{rel}}(X)$ is then shown to fit in the exact sequence:

$$(1)\qquad K_1(X) \xrightarrow{\mathcal{F}_*} K_1^{\mathrm{top}}(X) \xrightarrow{\varphi} K_0^{\mathrm{rel}}(X) \xrightarrow{\partial} K_0(X) \xrightarrow{\mathcal{F}_*} K_0^{\mathrm{top}}(X).$$

Secondly, let $F_{r(L)}$ (for $1\le r\le \dim X$) be the subspaces of the space $\wedge_{(L)}(X)$ of (logarithmic) $\mathcal{C}^\infty$ differential forms on X defining the Hodge filtration of the (logarithmic) de Rham complex of X (see Subsection 3.2, logarithmic forms correspond to the quasiprojective case). Chern-Simons transgression between two compatible connexions on E and F is shown (in Section 3) to provide characteristic class morphisms:

$$\mathcal{N}_P\colon K_0^{\mathrm{rel}}(X) \longrightarrow H^{2\bullet-1}(\wedge_{(L)}(X)/F_{\bullet(L)},d),$$
$$\mathcal{M}_P\colon K_0^{\mathrm{rel}}(X) \longrightarrow H^{2\bullet-1}(\wedge_{(L)}(X)/F_{\bullet(L)},d),$$

for an additive or multiplicative invariant polynomial P respectively (in the second line, the group structure of $H^{2\bullet-1}(\wedge(X)/F_\bullet,d)$ need not be the usual addition if X is a non Kähler or non compact complex analytic manifold). In particular for the total Chern class $P=c_{\mathrm{tot}}$

$$\mathcal{M}_{c_{\mathrm{tot}}}(E,F,f)=\overline{c_{\mathrm{tot}}}(\nabla_E,f^*\nabla_F)\wedge c_{\mathrm{tot}}^{-1}(\nabla_F^2),$$

where ∇_E and ∇_F are connexions on E and F compatible with their holomorphic (algebraic) structure, $c_{\mathrm{tot}}^{-1}(\nabla_F^2)$ is the Chern-Weil form calculated from the curvature of ∇_F and $\overline{c_{\mathrm{tot}}}(\nabla_E,f^*\nabla_F)$ is the Chern-Simons transgression form on X such that

$$d\,\overline{c_{\mathrm{tot}}}(\nabla_E,f^*\nabla_F)=c_{\mathrm{tot}}\big((f^*\nabla_F)^2\big)-c_{\mathrm{tot}}(\nabla_E^2).$$

Let $H^{2\bullet}_{\mathcal{D}}(X,\mathbb{Z}(\bullet))$ be Hodge-Deligne(-Beilinson) cohomology groups (see Subsection 3.2), the class $\mathcal{M}_{c_{\mathrm{tot}}}$ is proved to fit in the commutative diagram:

$$\begin{array}{ccccccc} K_1^{\mathrm{top}}(X) & \xrightarrow{\varphi} & K_0^{\mathrm{rel}}(X) & \xrightarrow{\partial} & K_0(X) & \xrightarrow{\mathcal{F}_*} & K_0^{\mathrm{top}}(X) \\ \downarrow{\scriptstyle Sc_{\mathrm{tot}}} & & \downarrow{\scriptstyle \mathcal{M}_{c_{\mathrm{tot}}}} & & \downarrow{\scriptstyle \tilde{c}_{\mathrm{tot}}} & & \downarrow{\scriptstyle c_{\mathrm{tot}}} \\ H^{\mathrm{odd}}(X,\mathbb{Z}) & \xrightarrow{\psi} & H^{2\bullet-1}\big(\wedge(X)/F_\bullet\big) & \xrightarrow{\partial_H} & H^{2\bullet}_{\mathcal{D}}\big(X,\mathbb{Z}(\bullet)\big) & \xrightarrow{f} & H^{\mathrm{even}}(X,\mathbb{Z}) \end{array} \tag{2}$$

(the second line is a classical exact sequence associated to Hodge-Deligne cohomology [17], Cor. 2.10 b).

An analogue construction is performed by Karoubi in [28], §6, for flat A-bundles, where A is a Banach algebra; this could be related to the theory developped here since the category of vector bundles on X has same algebraic K-theory groups as the ring of algebraic functions on some affine variety $\widetilde{X}$ related to X [26] (see [30], §4.1) but the constructions here are much nearer to Nadel's ideas and more adapted to study examples and direct images. However, Karoubi's multiplicative K-theory as treated in [30] fits nicely to the theory developped here as adding an intermediary line in the diagram above (see (29) below). The reason why higher K-theory is not studied here is that higher algebraic K-theory groups are difficult to describe explicitly (see [30], §4).

Nadel's results on his characteristic classes now generalise as follows:

The integrality property (see [35], §§6, 11 and 12) is recovered by the commutativity of the "left" square of the above diagram.

Let K_0^{def} be the subgroup of K_0^{rel} generated by deformations of holomorphic vector bundles, (see definition 5.1) and $F_{r-1}H^{2r-1}(\wedge(X)/F_r)$ the subgroup of classes in $H^{2r-1}(\wedge(X)/F_r)$ represented by differential forms lying in F_{r-1}, the rigidity result of Nadel (see [35], §§5, 10 and 12) reads now as follows: the image of $K_0^{\mathrm{def}}(X)$ under $\mathcal{N}_P$ or $\mathcal{M}_P$ (for any P) is included in $F_{\bullet-1}H^{2\bullet-1}(\wedge(X)/F_\bullet)$ (this equals $\bigoplus_p H^{p,p+1}(X)$ if X is compact Kähler), this result allows to precise Nadel's "topological monodromy restrictions"; these restrictions are generalised to higher order topological K-theory groups of X in Remark 5.7 below, and the rigidity result is generalised in Proposition 5.6 to reprove a statement by Esnault and Srinivas [16], Prop. 1, about vector bundles with holomorphic connexions.

For $1 \leq r \leq \dim X$, let $J_r(X)$ denote Griffiths' r-th intermediate jacobian

$$J_r(X) := H^{2r-1}\big(\wedge(X)/F_r, d)/\psi(H^{2r-1}(X,\mathbb{Z})\big)$$

define

$$K_0^{\mathrm{hom}} = \mathrm{Ker}\,(K_0 \xrightarrow{\mathcal{F}_*} K_0^{\mathrm{top}}) \cong K_0^{\mathrm{rel}}/\varphi(K_1^{\mathrm{top}})$$

and call K_0^{cont} the trace of K_0^{def} in K_0^{hom}. Then any holomorphic bundle on $\mathcal{M} \times X$ where $\mathcal{M}$ is any pointed complex manifold gives rise to a map

$$\mathcal{M} \longrightarrow K_0^{\mathrm{cont}} \xrightarrow{\mathcal{M}_{c_{\mathrm{tot}}}} J_r(X)$$

this map is proved to be holomorphic in §5.3 below, and its degree one part is (as Nadel proves it in [35], §9) the classical map $\mathcal{M} \to J_1(X)$ associated to the determinant line bundle. This leads to the question of the comparison of this map with the Albanese map of moduli spaces of vector bundles on X. The examples of abelian surfaces and of the cubic threefold are studied in §§6.5 and 5.3.

For X projective, let $\mathrm{CH}(X)$ denote the Chow ring of algebraic cycles in X modulo rational equivalence, and $\mathrm{CH}_{\mathrm{hom}}(X)$ its subgroup consisting of cycles homologically equivalent to zero, *i.e.* $\mathrm{CH}_{\mathrm{hom}}(X)$ is the kernel of the cycle map $\mathrm{CH}(X) \to H^{\mathrm{even}}(X, \mathbb{Z})$. The Chern isomorphism

$$K_0(X) \otimes \mathbb{Q} \cong \mathrm{CH}(X) \otimes \mathbb{Q}$$

(induced by the cycle valued Chern character) makes $K_0^{\mathrm{hom}}(X) \otimes \mathbb{Q}$ correspond to $CH_{\mathrm{hom}}(X) \otimes \mathbb{Q}$. Let

$$AJ\colon CH_{\mathrm{hom}}(X) \longrightarrow \bigoplus_r J_r(X)$$

denote the Abel-Jacobi map (see for example [31], Lecture 12). The conjecture of Nadel [35], §13, is then a consequence of the following

THEOREM 1.1. — *If X is projective, the following diagram commutes*

$$\begin{array}{ccc} K_0^{\mathrm{hom}}(X) & \xrightarrow{c_{\mathrm{tot}}-1} & CH_{\mathrm{hom}}(X) \\ {\scriptstyle \mathcal{M}_{c_{\mathrm{tot}}}}\Big\downarrow & & \Big\downarrow{\scriptstyle AJ} \\ \bigoplus_{r=1}^{\dim X} J_r(X) & =\!=\!= & \bigoplus_{r=1}^{\dim X} J_r(X) \end{array}$$

where 1 *denotes the unit* (*of intersection*) *in the Chow ring* $\mathrm{CH}(X)$ *and* c_{tot} *is the cycle valued total Chern class* [23].

This result means that $\mathcal{M}_{c_{\mathrm{tot}}}$ provides an analogue of the Abel-Jacobi map on K_0^{hom} (which can be defined on nonprojective manifolds). In particular the above question about the Albanese map of moduli spaces is closely related to the conjectural universality of the Abel-Jacobi map as a map from $\mathrm{CH}_{\mathrm{hom}}$ to Abelian varieties.

This theorem is proved in Section 4: the compatibility of Chern classes with the cycle class in Deligne-Beilinson cohomology (see [17], §8), and between the latter cycle class with the Abel-Jacobi map (see [19], [15], [17], §7.12, [25], Lemma 1.22 and [21], Thm. 3.5) being granted, this theorem becomes a consequence of the commutativity of the “central” square of diagram (2) above; in fact once Nadel's classes have been recognized to consist of Chern-Simons transgression forms, the compatibility between Chern classes in Cheeger-Simons characters and in Hodge-Deligne-Beilinson cohomology as studied by Brylinski [11], Zucker [39], Karoubi [30] and Gillet-Soulé [21] gives the answer.

Finally in Section 6, direct image questions are considered for relative K-theory: firstly, a morphism $K_0^{\mathrm{rel}}(X) \to K_0^{\mathrm{rel}}(Y)$ is constructed for any proper submersion $\pi\colon X \to Y$ of quasiprojective varieties. This needs resolutions to the right in $K_0^{\mathrm{rel}}(X)$ by triples (A, B, g) where A and B are π-acyclic in the sense that $R^j\pi_*A$ and $R^j\pi_*B$ vanish for all $j \geq 1$. A trick from [2] (Prop. 2.2, see also [3], §9) is then needed to define the direct image of such a (A, B, g) because the $\overline{\partial} + \overline{\partial}^*$ operators associated to connexions on A realising some homotopy between two compatible connexions on A and B respectively need not have kernels of constant dimension. A Riemann-Roch type theorem for $\mathcal{N}_{\mathrm{ch}}$ is then proved in the projective case: it is obtained by "integrating along $[0, 1]$" the refinement (at the level of differential forms) of the families index theorem on $X \times [0, 1]$ taken for the $\overline{\partial} + \overline{\partial}^*$ operator on vector bundles which may be nonholomorphic. On $X \times \{0\}$ and $X \times \{1\}$ however, the vector bundles are holomorphic and the double transgression of the families index theorem of Bismut and Köhler [9] plays a crucial role. This Riemann-Roch theorem is then applied to recognize the map $\mathcal{M}_{c_{\mathrm{tot}}}$ in Yoshioka's construction of the Albanese map for moduli spaces of vector bundles on Abelian surfaces [38], [37].

Secondly for a closed immersion of projective varieties $\iota\colon Y \to X$, the direct image morphism is made available only from $K_0^{\mathrm{def}}(Y)$ to $K_0^{\mathrm{def}}(X)$; this is because $\mathcal{C}^\infty$ isomorphisms do not fit nicely with with resolutions of vector bundles on Y by vector bundles on X. The Riemann-Roch statement is obtained from Bismut Gillet and Soulé's double transgression for immersions [8]. Note that these Riemann-Roch results are not consequences of the "usual" Riemann-Roch theorem on Deligne-Beilinson cohomology proved by Gillet [18] because Gillet's result cannot take more in consideration than $K_0^{\mathrm{hom}}(X)$ in the case of closed immersions and $K_0^{\mathrm{hom}}(X) \otimes \mathbb{Q}$ in the general case.

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2. Relative K-theory

Two cases are considered here: X is either a smooth quasiprojective complex manifold or a complex analytic manifold. The results will be stated in the first case (the second case is deduced by replacing everywhere "algebraic" by "holomorphic"). In 2.1, the "relative" K-theory group $K_0^{\mathrm{rel}}(X)$ is defined, it