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AN APPROXIMATION PROPERTY OF QUADRATIC IRRATIONALS

BY TAKAO KOMATSU

Dedicated to Professor Iekata Shiokawa
on the occasion of his 60th birthday

ABSTRACT. — Let $\alpha > 1$ be irrational. Several authors studied the numbers

$$\ell^m(\alpha) = \inf\{\,|y| : y \in \Lambda_m,\ y \neq 0\},$$

where m is a positive integer and Λ_m denotes the set of all real numbers of the form $y = \epsilon_0\alpha^n + \epsilon_1\alpha^{n-1} + \cdots + \epsilon_{n-1}\alpha + \epsilon_n$ with restricted integer coefficients $|\epsilon_i| \leq m$. The value of $\ell^1(\alpha)$ was determined for many particular Pisot numbers and $\ell^m(\alpha)$ for the golden number. In this paper the value of $\ell^m(\alpha)$ is determined for irrational numbers α, satisfying $\alpha^2 = a\alpha \pm 1$ with a positive integer a.

RÉSUMÉ (*Une approximation des irrationnels quadratiques*). — Soit $\alpha > 1$ un irrationnel. Plusieurs auteurs ont étudié les nombres

$$\ell^m(\alpha) = \inf\{\,|y| : y \in \Lambda_m,\ y \neq 0\},$$

où m est un entier positif et Λ_m est l'ensemble de tous les réels de la forme $y = \epsilon_0\alpha^n + \epsilon_1\alpha^{n-1} + \cdots + \epsilon_{n-1}\alpha + \epsilon_n$ avec des $|\epsilon_i| \leq m$ entiers. La valeur de $\ell^1(\alpha)$ a été précisée pour beaucoup de nombres de Pisot et $\ell^m(\alpha)$ pour le nombre d'or. Dans cet article, on détermine $\ell^m(\alpha)$ lorsque α est un irrationnel qui satisfait $\alpha^2 = a\alpha \pm 1$ avec a entier positif.

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1. Introduction

Let α be a positive real number and an integer $m \geq 1$. Denote by $\Lambda = \Lambda_m$ the set of all real numbers y having at least one representation of the form

$$y = \epsilon_0\alpha^n + \epsilon_1\alpha^{n-1} + \cdots + \epsilon_{n-1}\alpha + \epsilon_n$$

with some positive integer n and $|\epsilon_i| \leq m$, $\epsilon_i \in \mathbb{Z}$ $(i = 0, 1, \ldots, n)$. Set

$$\ell^m(\alpha) = \inf\bigl\{|y| : y \in \Lambda_m,\ y \neq 0\bigr\}.$$

Several authors studied the numbers $\ell^m(\alpha)$. The value of $\ell^1(\alpha)$ was determined for many particular Pisot numbers (see [1], [3], [4], [5], [6], [7], [8]) and $\ell^m(\alpha)$ for the golden number (see [8]). In this paper the value of $\ell^m(\alpha)$ is determined for two kinds of irrational numbers:

$$\alpha = \frac{1}{2}\bigl(a + \sqrt{a^2+4}\,\bigr) = [a; a, a, \ldots] \quad (a \geq 1) \quad \text{and}$$

$$\alpha = \frac{1}{2}\bigl(a + \sqrt{a^2-4}\,\bigr) = [a-1; 1, a-2, 1, a-2, \ldots] \quad (a \geq 3).$$

We shall prove the following two theorems.

THEOREM 1.1. — *Let* $\alpha = \frac{1}{2}(a + \sqrt{a^2+4}\,)$ *$(a \geq 1)$. If*

$$\alpha^k(\alpha - 1) < m \leq \alpha^{k+1}(\alpha - 1)$$

for some integer $k \geq -1$*, then*

$$\ell^m(\alpha) = \bigl|q_k\alpha - p_k\bigr|.$$

THEOREM 1.2. — *Let* $\alpha = \frac{1}{2}(a + \sqrt{a^2-4})$ $\quad (a \geq 3)$.

- *If for some non-negative integer* i*,*

$$\alpha^i(\alpha - a + 1) < m \leq \alpha^i(a - 2)$$

then

$$\ell^m(\alpha) = \bigl|q_{2i-1}\alpha - p_{2i-1}\bigr|.$$

- *If for some non-negative integer* i*,*

$$\alpha^i(a - 2) < m < \alpha^{i+1}(\alpha - a + 1)$$

then

$$\ell^m(\alpha) = |q_{2i}\alpha - p_{2i}|.$$

In addition to prove these two theorems, we shall show how to find a representation form $y = \epsilon_0\alpha^n + \epsilon_1\alpha^{n-1} + \cdots + \epsilon_{n-1}\alpha + \epsilon_n$ which gives $\ell^m(\alpha) = |q_k\alpha - p_k|$.

2. General sketch

If α is a root of the quadratic equation, then any of the form

$$\epsilon_0\alpha^n + \epsilon_1\alpha^{n-1} + \cdots + \epsilon_{n-1}\alpha + \epsilon_n \quad (n \geq 2)$$

can be reduced to the form $q\alpha - p$ for some integers q and p. We can set $\alpha > 1$. For, $\ell^m(\alpha) = 0$ if $0 < \alpha < 1$; $\ell^m(\alpha) = 1$ if $\alpha = 1$. Concerning the linear form $q\alpha - p$, the following approximation theorem is well-known (see Thm. 5E (ii) in [10], *e.g.*).

THEOREM A. — *If $k \geq 1$, $0 < q \leq q_k$ and $p/q \neq p_k/q_k$, $p/q \neq p_{k-1}/q_{k-1}$, then*

$$|q_{k-1}\alpha - p_{k-1}| < |q\alpha - p|.$$

Here, $p_k/q_k = [a_0; a_1, \ldots, a_k]$ denotes the k-th convergent of the continued fraction expansion of α, $\alpha = [a_0; a_1, a_2, \ldots]$. Namely,

$$\alpha = a_0 + \frac{1}{\alpha_1}, \qquad a_0 = \lfloor \alpha \rfloor,$$

$$\alpha_n = a_n + \frac{1}{\alpha_{n+1}}, \qquad a_n = \lfloor \alpha_n \rfloor \quad (n \geq 1)$$

and

$$p_k = a_k p_{k-1} + p_{k-2} \quad (k \geq 0), \qquad p_{-1} = 1, \qquad p_{-2} = 0,$$

$$q_k = a_k q_{k-1} + q_{k-2} \quad (k \geq 0), \qquad q_{-1} = 0, \qquad q_{-2} = 1.$$

Therefore, this kind of problems is equivalent to how to find the least m with $|\epsilon_i| \leq m$, $\epsilon_i \in \mathbb{Z}$ $(0 \leq i \leq n)$, satisfying

$$\epsilon_0\alpha^n + \epsilon_1\alpha^{n-1} + \cdots + \epsilon_{n-1}\alpha + \epsilon_n = q\alpha - p = \pm(q_k\alpha - p_k)$$

for a fixed integer k. In other words, $\ell^m(\alpha)$ is equal a priori to some $|q_k\alpha - p_k|$ for every m. Namely, for any positive integer m there exists an integer k such that for some $y = y_{-1}, y_0, \ldots, y_k$ we have

$$y_{-1} = |q_{-1}\alpha - p_{-1}|, \quad y_0 = |q_0\alpha - p_0|, \ \ldots, \quad y_k = |q_k\alpha - p_k|,$$

but $y \neq |q_{k+1}\alpha - p_{k+1}|$ for any y. From the result of van Ravenstein [9], the integer q satisfying $\ell^m(\alpha) = q\alpha - p$ should be one of the first values

$$\big\{(-1)^n j q_n/q_{n+1}\big\} q_{n+1} \quad (j = 1, 2, \ldots),$$

and the integer p be its counterpart. Notice that larger m becomes, more choices each ϵ_i can have. So, always $\ell^m(\alpha) \leq \ell^{m+1}(\alpha)$ for every m. If $q > q_{k+1}$ then from our decision of m (see (7) below) we could choose some y so that $y = |q_{k+1}\alpha - p_{k+1}|$ because

$$|q|s_{n-1} + |p|s_n > q_{k+1}s_{n-1} + p_{k+1}s_n.$$

Hence, it is sufficient to consider the integers q with $q_k < |q| < q_{k+1}$. But, by Theorem A always $|q\alpha - p| > |q_k\alpha - p_k|$ holds for such q's.

Suppose that α is the larger root of the quadratic equation $x^2 = ax + b$. Here, $a, b \in \mathbb{Z}$ because both q and p are integers. Notice that $x^2 - ax - b = (x-\alpha)(x-\beta)$, where

$$\alpha = \frac{a + \sqrt{a^2+4b}}{2} \quad \text{and} \quad \beta = \frac{a - \sqrt{a^2+4b}}{2},$$

satisfying $\alpha + \beta = a$, $\alpha\beta = -b$. By $\alpha > 1$, we have $a + b > 1$, satisfying $a^2 + 4b > 0$. Put

$$\alpha^n = s_n\alpha + t_n \quad \text{for} \quad n \geq 0.$$

Lemma 2.1. — *One has*

$$s_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad t_n = b\frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta} = bs_{n-1}. \tag{1}$$

$$s_{n-1}s_i - s_ns_{i-1} = (-b)^{i-1}s_{n-i} \quad (i = 1, 2, \ldots, n). \tag{2}$$

Proof. — (1) Since the recurrence relation $r_n = ar_{n-1} + br_{n-2}$ has the general solution

$$r_n = \frac{(\alpha^{n-1} - \beta^{n-1})r_2 - \alpha\beta(\alpha^{n-2} - \beta^{n-2})r_1}{\alpha - \beta},$$

by using $s_2 = a$, $s_1 = 1$, $t_2 = b$ and $t_1 = 0$ we have

$$s_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad t_n = b\frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta} = bs_{n-1}.$$

(2) For $i = 1, 2, \ldots, n$

$$\begin{aligned} s_{n-1}s_i - s_ns_{i-1} &= s_{n-1}(as_{i-1} + bs_{i-2}) - (as_{n-1} + bs_{n-2})s_{i-1} \\ &= -b(s_{n-2}s_{i-1} - s_{n-1}s_{i-2}) \\ &= (-b)^2(s_{n-3}s_{i-2} - s_{n-2}s_{i-3}) = \cdots \\ &= (-b)^{i-1}(s_{n-i}s_1 - s_{n-i+1}s_0) \\ &= (-b)^{i-1}s_{n-i}. \end{aligned}$$

□

By using s_n, y can be written as a linear form:

$$\begin{aligned} y &= \epsilon_0\alpha^n + \epsilon_1\alpha^{n-1} + \cdots + \epsilon_{n-1}\alpha + \epsilon_n \\ &= (\epsilon_0 s_n + \epsilon_1 s_{n-1} + \cdots + \epsilon_{n-2}s_2 + \epsilon_{n-1}s_1)\alpha \\ &\qquad + b(\epsilon_0 s_{n-1} + \epsilon_1 s_{n-2} + \cdots + \epsilon_{n-2}s_1) + \epsilon_n. \end{aligned}$$

Suppose that

$$\epsilon_0 s_n + \epsilon_1 s_{n-1} + \cdots + \epsilon_{n-2} s_2 + \epsilon_{n-1} s_1 = q_k, \tag{3}$$

$$b(\epsilon_0 s_{n-1} + \epsilon_1 s_{n-2} + \cdots + \epsilon_{n-2} s_1) + \epsilon_n = -p_k \tag{4}$$

for some integer k. Otherwise, we interchange each sign of ϵ_i ($i = 0, 1, \ldots, n$). We shall find the least integer m (say, m') which satisfies $|\epsilon_i| \le m$ for all $i = 0, 1, \ldots, n$. By eliminating ϵ_0 we have

$$\begin{aligned} &b(s_{n-1}^2 - s_n s_{n-2})\epsilon_1 + b(s_{n-1}s_{n-2} - s_n s_{n-3})\epsilon_2 \\ &\quad + \cdots + b(s_{n-1}s_2 - s_n s_1)\epsilon_{n-2} + b s_{n-1}\epsilon_{n-1} - s_n \epsilon_n = q_k b s_{n-1} + p_k s_n. \end{aligned}$$

By Lemma 2.1(2), we obtain

$$\begin{aligned} &-(-b)^{n-1} s_1 \epsilon_1 - (-b)^{n-2} s_2 \epsilon_2 - \cdots - (-b)^2 s_{n-2}\epsilon_{n-2} \\ &\qquad + b s_{n-1}\epsilon_{n-1} - s_n \epsilon_n = q_k b s_{n-1} + p_k s_n. \end{aligned} \tag{5}$$

If $\gcd(a, b) = 1$, then we have $\gcd(s_{i+1}, b s_i) = 1$ ($i \ge 1$), yielding

$$\gcd(b^{n-1}s_1, b^{n-2}s_2, \ldots, b s_{n-1}, s_n) = 1.$$

In fact, $\gcd(s_2, b s_1) = \gcd(a, b) = 1$. Assume that $\gcd(s_n, b s_{n-1}) = 1$ for some n. Suppose that, however,

$$\gcd(s_{n+1}, b s_n) = \gcd(a s_n + b s_{n-1}, b s_n) = c$$

with $c \ge 2$. Since $c \mid b s_n$, we have for some divisor of c, say $c_1 > 1$, $c_1 \mid b$ or $c_1 \mid s_n$. If $c_1 \mid b$ then by $c_1 \mid s_{n+1}$ and $c_1 \nmid a$ we have $c_1 \mid s_n$, yielding $\gcd(s_n, b s_{n-1}) = c_1$. If $c_1 \mid s_n$ then by $c_1 \mid s_{n+1}$ we have $c_1 \mid b s_{n-1}$, yielding $\gcd(s_n, b s_{n-1}) = c_1$, which is the contradiction again.

Therefore, the linear equation (5) is solvable in integers. We shall show the concrete step to obtain one of solutions in (5) in the following sections. $|\epsilon_n|$, $|\epsilon_{n-1}|, \ldots, |\epsilon_1|$ can be chosen as lexicographically minimal among those giving $\ell^m(\alpha) = |q_k \alpha - p_k|$.

After choosing the integers from ϵ_n to ϵ_1, ϵ_0 can be naturally determined as an integer if $\gcd(a, b) = 1$. For, by (3) and (4)

$$\epsilon_0 = \frac{q_k - (\epsilon_1 s_{n-1} + \cdots + \epsilon_{n-1} s_1)}{s_n} = \frac{-p_k - b(\epsilon_1 s_{n-2} + \cdots + \epsilon_{n-2} s_1) - \epsilon_n}{b s_{n-1}}.$$

Since both of two fractions are integral and $\gcd(s_n, b s_{n-1}) = 1$, both of fractions cannot be the same unless ϵ_0 becomes integral.

We assume that $b = \pm 1$ in stating two theorems. This assumption guarantees that ϵ_1 becomes integral after deciding $\epsilon_n, \epsilon_{n-1}, \ldots, \epsilon_2$. Otherwise, ϵ_1 may not become integral by the method in this paper.