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INVARIANCE OF GLOBAL SOLUTIONS OF THE HAMILTON-JACOBI EQUATION

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Abstract. — We show that every global viscosity solution of the Hamilton-Jacobi equation associated with a convex and superlinear Hamiltonian on the cotangent bundle of a closed manifold is necessarily invariant under the identity component of the group of symmetries of the Hamiltonian (we prove that this group is a compact Lie group). In particular, every Lagrangian section invariant under the Hamiltonian flow is also invariant under this group.

Résumé (*Invariance des solutions globales de l'équation de Hamilton-Jacobi*)
On prouve que toute solution globale de viscosité de l'équation de Hamilton-Jacobi associée à un hamiltonien convexe et superlinéaire sur le fibré cotangent d'une variété fermée est toujours invariante sous l'action de la composante neutre du groupe de symétries du hamiltonien (on montre que ce groupe est un groupe de Lie compact). En particulier, toute section lagrangienne du fibré cotangent qui est preservée par le flot hamiltonien doit être invariante sous cette action.

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1. Introduction

Let M be a closed manifold, and let $H : T^*M \to \mathbb{R}$ be a $\mathcal{C}^\infty$ Hamiltonian that is $\mathcal{C}^2$-strictly convex and superlinear on the fibers of the cotangent bundle $\pi^* : T^*M \to M$.

In [3], generalizing work by Lions, Papanicolau & Varadhan, Fathi proved the existence of global viscosity solutions, also called weak KAM solutions, of the Hamilton-Jacobi equation

$$H(x, \mathrm{d}_x u) = c$$

and that these solutions only exist for the value $c = c(L)$, which equals Mañé's critical value of the associated Lagrangian. The latter can also be characterized in terms of Mather's minimizing measures (see [7], [8]). The solutions are given modulo constants by the fixed points of the Lax-Oleinik semigroups T_t^- and T_t^+ (see below for the definition of T_t^- and T_t^+). Now let $\mathcal{S}_-$ and $\mathcal{S}_+$ be the set of weak KAM solutions of T_t^- and T_t^+ respectively. One has that $\mathcal{S}_- \cap \mathcal{S}_+ = \mathcal{S}$, the set of classical solutions of the Hamilton-Jacobi equation, *i.e.* of class $\mathcal{C}^1$.

The weak KAM solutions are very useful in the study of the dynamics of the Hamiltonian vector field X_H associated with H (see also [1], [2]).

We will denote by Γ_H the group of diffeomorphisms of M of class $\mathcal{C}^1$ that preserve H, more precisely

$$\Gamma_H = \big\{g \in \mathrm{Diff}^1(M)\,;\, H(g(x), p) = H(x, p \circ \mathrm{d}_x g),\ \forall x \in M,\ p \in T^*_{g(x)}M\big\}$$

endowed with the topology of uniform convergence. Let Γ_H^0 be the identity component of Γ_H. We shall prove in Section 4 that Γ_H is a compact Lie group.

In [5], the proof of the existence of weak KAM solutions is generalized to the case when M is not necesarily compact, with the additional hypothesis of uniform superlinearity (with respect to a complete Riemannian metric) of the Hamiltonian and its associated Lagrangian. In [5], we also show the existence of Γ_H-invariant weak KAM solutions for values of the constant greater or equal than a certain value $c_{\mathrm{inv}} \geq c(L)$. It follows that if M is compact $c_{\mathrm{inv}} = c(L)$. We will prove later these facts in a slightly simplified way using compactness.

On the other hand, if M is not compact the inequality $c_{\mathrm{inv}} \geq c(L)$ could be strict. This follows from the examples given by G. & M. Paternain [10] on the universal cover of closed surface of genus 2.

In this paper we show:

Theorem 1. — *Let M be a closed manifold, and let $H : T^*M \to \mathbb{R}$ be a $\mathcal{C}^\infty$ Hamiltonian that is convex and superlinear on the fibers of the cotangent bundle of M. If Γ_H is the symmetry group of H, then every weak KAM solution of H is Γ_H^0-invariant, where Γ_H^0 denote the identity component of Γ_H.*

In general, Hamiltonians have trivial symmetry groups like general Riemannian metrics which usually have trivial isometry groups. However we find

Hamiltonian systems with symmetries quite often in the applications, and these symmetries are very useful for a detailed study of the system. If the dimension of Γ_H is sufficiently large we can find all weak KAM solutions by integration, as we will see in the case when H is the Hamiltonian of the mechanical system determined by the motion of a particle on the n-sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ under the effect of a potential $U(x) = x_{n+1}$. In this case we can reduce the problem to finding the weak KAM solutions of the pendulum on the circle.

In particular, our result applies to the solutions of the Hamilton-Jacobi equation of class $\mathcal{C}^2$ that correspond to exact Lagrangian sections of T^*M invariant under the Hamiltonian flow associated with H. To reduce the study of Lagrangian sections to exact ones, we shall recall in Section 5 that given any cohomology class in $H^1(M,\mathbb{R})$ there exist a closed Γ_H-invariant 1-form that represent the class. Combining this result with theorem 1 we obtain the following corollary whose proof will also be given in Section 5:

Corollary 2. — *Every Lagrangian section of T^*M invariant under the Hamiltonian flow of H is also invariant under Γ_H^0.*

2. Weak KAM solutions and Mather's set

Before giving the proof of Theorem 1, we briefly recall the properties of the weak KAM solutions which we will use. The details of the proofs can be found in [3] and [4].

Let us introduce initially the Lagrangian corresponding to H like its convex dual on the tangent bundle of M:

$$L : TM \longrightarrow \mathbb{R}, \quad L(x,v) = \sup\big\{p(v) - H(x,p)\,;\, p \in T_x^*M\big\}.$$

It is well-known that L is also of $\mathcal{C}^\infty$ class, strictly convex and superlinear on the fibers, *i.e.* its second derivative $\partial^2 L/\partial v^2$ is definite positive everywhere and for all $K \in \mathbb{R}$ there exists a constant C_K such that

$$\forall (x,v) \in TM, \quad L(x,v) \geq K\|v\| + C_K.$$

The Legendre transform $\mathcal{L} : TM \to T^*M$,

$$\mathcal{L}(x,v) = \Big(x, \frac{\partial L}{\partial v}(x,v)\Big),$$

is a diffeomorphism which conjugates the Euler-Lagrange flow defined by L on M, which is denoted ϕ_t^L, with the Hamiltonian flow of H.

The action of L on a piecewise $\mathcal{C}^1$ curve $\gamma : [a,b] \to M$ is as usual

$$A_L(\gamma) = \int_a^b L\big(\gamma(s), \dot\gamma(s)\big)\,\mathrm{d}s.$$

We will say that a function $u : M \to \mathbb{R}$ is *dominated* by $L + c$ (for certain value of $c \in \mathbb{R}$) and we will write

$$u \prec L + c$$

if for each piecewise $\mathcal{C}^1$ curve $\gamma : [a, b] \to M$ we have

$$u\big(\gamma(b)\big) - u\big(\gamma(a)\big) \le A_L(\gamma) + c(b - a).$$

From the superlinearity of L it is easy to deduce that dominated functions are Lipschitz, with a Lipschitz constant which only depends, once fixed the metric on M, on the constant c and the Lagrangian; in particular, in accordance with the Rademacher's theorem (see [11]), they are differentiable almost everywhere.

The main reason we are interested in dominated functions is that they constitute a suitable space where Lax-Oleinik's semigroups of operators can be studied. In this way, we will obtain weak KAM solutions. Since we already know that the solutions are dominated by $L + c(L)$, where $c(L)$ is the critical value of L, we can directly introduce the space

$$\mathbb{H} = \big\{u \in \mathcal{C}^0(M, \mathbb{R})\, ;\, u \prec L + c(L)\big\}.$$

On this space we can define the non linear operators $u \mapsto T_t^- u$, $u \mapsto T_t^+ u$ for each $t \ge 0$

$$T_t^- u(x) = \inf_{\gamma\in\mathcal{C}^-} \big\{u(\gamma(0)) + A_L(\gamma)\big\}, \quad T_t^+ u(x) = \sup_{\gamma\in\mathcal{C}^+} \big\{u(\gamma(t)) - A_L(\gamma)\big\},$$

where

$$\begin{aligned}\mathcal{C}^- &= \big\{\gamma : [0, t] \to M\, ;\, \text{piecewise } \mathcal{C}^1 \text{ with } \gamma(t) = x\big\},\\ \mathcal{C}^+ &= \big\{\gamma : [0, t] \to M\, ;\, \text{piecewise } \mathcal{C}^1 \text{ with } \gamma(0) = x\big\}.\end{aligned}$$

From the definition it follows the semigroup property

$$T_t^- \circ T_s^- = T_{t+s}^- \quad \text{for all } t, s \ge 0,$$

and that

$$T_t^-(u + c) = T_t^-(u) + c \quad \text{for all } c \in \mathbb{R}.$$

On the other hand, it is clear that $u + c \in \mathbb{H}$ whenever $u \in \mathbb{H}$; this allows us to define the quotient semigroup $\widehat{T}_t^-$ acting on $\widehat{\mathbb{H}}$, the quotient set of $\mathbb{H}$ by the space of constant functions, by

$$\widehat{T}_t^-[u] = [T_t^- u].$$

Analogously, we can define the quotient semigroup $\widehat{T}_t^+$.

Definition 3 (Weak KAM solution). — We say that $u \in \mathbb{H}$ is a *global viscosity solution* of the Hamilton-Jacobi equation, or a *weak KAM solution*, if

$$\widehat{T}_t^-[u] = [u] \quad \text{or} \quad \widehat{T}_t^+[u] = [u] \quad \text{for all } t \in \mathbb{R}.$$

We call $\mathcal{S}_-$ and $\mathcal{S}_+$ respectively the sets defined by the above relations.

The existence of these solutions is obtained in [3] through the application of the fixed point theorem of Schauder and Tykhonov; this requires to show the continuity of the semigroups and the compactness of the convex $\widehat{\mathbb{H}}$. In the same article, it is shown that the relations which define $\mathcal{S}_-$ and $\mathcal{S}_+$ sets, are equivalent to

$$T_t^- u = u - c(L)tQ \quad \text{and} \quad T_t^+ u = u + c(L)t, \quad \forall t \geq 0,$$

and that weak KAM solutions verify the Hamilton-Jacobi equation at every point where they are differentiable.

Weak KAM solutions are also characterized by the fact of being dominated by $L + c(L)$ and by the existence of certain curves on which their variation is maximal; more precisely,

Proposition 4 (Fathi [3]). — *A function $u : M \to \mathbb{R}$ is in $\mathcal{S}_-$ if and only if:*

a) *$u \prec L + c(L)$, where $c(L)$ is the critical value of L,*

b) *for all $x \in M$ there exists an extremal of L, $\gamma_x : (-\infty, 0] \to M$ with $\gamma_x(0) = x$, and such that for all $t \geq 0$ we have*

$$u(x) - u\big(\gamma_x(-t)\big) = \int_{-t}^{0} L\big(\gamma_x(s), \dot{\gamma}_x(s)\big)\,\mathrm{d}s + c(L)t.$$

Moreover, the set of differentiability points of a function satisfying a) *contains the points $x \in M$ for which there exists $\epsilon > 0$ and an extremal $\gamma : [-\epsilon, \epsilon] \to M$, such that $\gamma(0) = x$ and*

$$u\big(\gamma(\epsilon)\big) - u\big(\gamma(-\epsilon)\big) = \int_{-\epsilon}^{\epsilon} L\big(\gamma(s), \dot{\gamma}(s)\big)\,\mathrm{d}s + 2\epsilon\, c(L).$$

The characterization of the functions in $\mathcal{S}_+$ is analogous, it is enough to replace the curves of the condition b) by curves of the form $\gamma_x : [0, +\infty) \to M$ with $\gamma_x(0) = x$ along which the equality is satisfied.

Let now μ be a Borel measure on TM, invariant under the Euler-Lagrange flow, and let $u : M \to \mathbb{R}$ be a $(L+c)$-dominated function. Because of invariance of μ we have

$$\int_{TM} (u \circ \pi \circ \phi_1^L - u \circ \pi)\,\mathrm{d}\mu = 0,$$

where $\pi : TM \to M$ is the canonical projection of the tangent bundle. If one applies for each $v \in TM$, the domination of u by $L+c$ to the curve $t \mapsto \pi \circ \phi_s^L(v)$ with t varying in $[0, 1]$, it results from it that

$$c + \int_{TM} \int_0^1 L \circ \phi_s^L \,\mathrm{d}s\,\mathrm{d}\mu \geq 0.$$