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## INVOLUTIVITY OF TRUNCATED MICROSUPPORTS

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ABSTRACT. — Using a result of J.-M. Bony, we prove the weak involutivity of truncated microsupports. More precisely, given a sheaf F on a real manifold and  $k \in \mathbb{Z}$ , if two functions vanish on  $SS_k(F)$ , then so does their Poisson bracket.

RÉSUMÉ (Involutivité des micro-supports tronqués). — Utilisant un résultat de J.-M. Bony, nous prouvons l'involutivité faible des micro-supports tronqués. Plus précisément, étant donnés un faisceau F sur une variété réelle et  $k \in \mathbb{Z}$ , si deux fonctions s'annulent sur  $SS_k(F)$ , il en est de même de leur crochet de Poisson.

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## 1. Introduction

The notion of microsupport of sheaves was introduced by two of the present authors (M.K. and P.S.) in the course of the study of the theory of linear partial differential equations. References are made to [3]. These authors also introduced a variant of this notion, that of "truncated microsupport" and developed its study with the third author in [2].

A crucial result in the microlocal theory of sheaves is the involutivity of the microsupport. This property does not hold true any more for the truncated microsupport, but a weak form of it does. More precisely, we prove here that the truncated microsupport is stable by Poisson bracket, that is, if two functions vanish on it, so does their Poisson bracket. The main technical tool is a result of J.-M. Bony which asserts the same property for the normal cone to a closed subset.

## 2. Notations and review

We will follow the notations of [3] and [2]. For the reader convenience, we recall some of them as well as the definition of the truncated microsupport. Let X be a real analytic manifold. We denote by  $\tau: TX \longrightarrow X$  the tangent bundle to X and by  $\pi: T^*X \longrightarrow X$  the cotangent bundle. For a smooth submanifold Y of X,  $T_YX$  denotes the normal bundle to Y and  $T_Y^*X$  the conormal bundle. In particular,  $T_X^*X$  is the zero section of  $T^*X$ . We set  $\dot{T}^*X = T^*X \setminus X$ , and denote by  $\dot{\pi}: \dot{T}^*X \longrightarrow X$  the restriction of  $\pi$  to  $\dot{T}^*X$ .

For a morphism  $f: X \longrightarrow Y$  of real manifolds, we denote the associated morphisms by

$$f_{\pi} \colon X \times_Y T^*Y \longrightarrow T^*Y$$
 and  $f_d \colon X \times_Y T^*Y \longrightarrow T^*X$ .

For a subset A of  $T^*X$ , we denote by  $A^a$  the image of A by the antipodal map  $a: (x;\xi) \mapsto (x;-\xi)$ . The closure of A is denoted by  $\overline{A}$ .

For a cone  $\gamma \subset TX$ , the polar cone  $\gamma^{\circ}$  to  $\gamma$  is the convex cone in  $T^*X$  defined by

$$\gamma^{\circ} = \{ (x;\xi) \in T^*X; x \in \pi(\gamma) \text{ and } \langle v,\xi \rangle \ge 0 \text{ for any } (x;v) \in \gamma \}.$$

A closed convex cone  $\gamma$  is called *proper* if  $0 \in \gamma$  and  $Int(\gamma^{\circ}) \neq \emptyset$ .

Let  $\boldsymbol{k}$  be a field. One denotes by  $\operatorname{Mod}(\boldsymbol{k}_X)$  the abelian category of sheaves of  $\boldsymbol{k}$ -vector spaces and by  $D^b(\boldsymbol{k}_X)$  its bounded derived category. One denotes by  $D^b_{\mathbb{R}-c}(\boldsymbol{k}_X)$  the full triangulated subcategory of  $D^b(\boldsymbol{k}_X)$  consisting of objects with  $\mathbb{R}$ -constructible cohomology. If X is a complex manifold, one denotes by  $D^b_{\mathbb{C}-c}(\boldsymbol{k}_X)$  the full triangulated subcategory of  $D^b(\boldsymbol{k}_X)$  consisting of objects with  $\mathbb{C}$ -constructible cohomology.

If S is a locally closed subset of X, one denotes by  $\mathbf{k}_{XS}$  the sheaf on X which is the constant sheaf with stalk  $\mathbf{k}$  on S and 0 on  $X \setminus S$ . If there is no risk of confusion, we may write  $\mathbf{k}_S$  instead of  $\mathbf{k}_{XS}$ .

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For  $k \in \mathbb{Z}$ , we denote as usual by  $D^{\geq k}(\mathbf{k}_X)$  (resp.  $D^{\leq k}(\mathbf{k}_X)$ ) the full additive subcategory of  $D^b(\mathbf{k}_X)$  consisting of objects F satisfying  $H^j(F) = 0$  for any j < k (resp.  $H^j(F) = 0$  for any j > k).

We denote by  $\tau^{\leq k} \colon D(\mathbf{k}_X) \longrightarrow D^{\leq k}(\mathbf{k}_X)$  the truncation functor.

Recall that for  $F \in D(\mathbf{k}_X)$  the morphism  $\tau^{\leq k}F \longrightarrow F$  induces isomorphisms  $H^j(\tau^{\leq k}F) \xrightarrow{\sim} H^j(F)$  for  $j \leq k$  and  $H^j(\tau^{\leq k}F) = 0$  for j > k.

If F is an object of  $D^b(\mathbf{k}_X)$ , SS(F) denotes its microsupport, a closed  $\mathbb{R}^+$ -conic subset of  $T^*X$ . For  $p \in T^*X$ ,  $D^b(\mathbf{k}_X;p)$  denotes the localization of  $D^b(\mathbf{k}_X)$  by the full triangulated subcategory consisting of objects F such that  $p \notin SS(F)$ . A property holds "microlocally on a subset S of  $T^*X$ " if it holds in the category  $D^b(\mathbf{k}_X;p)$  for any  $p \in S$ .

We recall the definition of involutivity of [3]. This notion makes use of that of "normal cone" (see [3, Def. 4.1.1]). For a pair of subsets  $S_1$  and  $S_2$  of a manifold X, the normal cone  $C_p(S_1, S_2)$  at  $p \in X$  is defined as follows: it is a closed cone in the tangent space  $T_pX$  consisting of points v such that, for a local coordinate system, there exist a sequence  $\{x_n\}_n$  in  $S_1$ ,  $\{y_n\}_n$  in  $S_2$  and a sequence  $\{a_n\}_n$  in  $\mathbb{R}_{\geq 0}$  such that  $x_n$  and  $y_n$  converge to p and  $a_n(x_n - y_n)$ converges to v. For a subset S,  $C_p(S, \{p\})$  is denoted by  $C_p(S)$ .

DEFINITION 2.1. — (see [3, Definition 6.5.1]) Let S be a locally closed subset of  $T^*X$  and let  $p \in S$ . One says that S is *involutive* at p if for any  $\theta \in T_p^*(T^*X)$  such that the normal cone  $C_p(S, S)$  is contained in the hyperplane  $\{v \in T_p(T^*X); \langle v, \theta \rangle = 0\}$  one has  $H(\theta) \in C_p(S)$ . Here  $H: T_p^*(T^*X) \xrightarrow{\sim} T_p(T^*X)$  is the Hamiltonian isomorphism.

If S is involutive at each  $p \in S$ , one says that S is *involutive*.

The involutivity theorem of [3, Theorem 6.5.4] asserts that the microsupport of sheaves is involutive.

The following definition was introduced by the authors of [3] and developed in [2].

DEFINITION 2.2. — Let X be a real analytic manifold and let  $p \in T^*X$ . Let  $F \in D^b(\mathbf{k}_X)$  and  $k \in \mathbb{Z}$ . The closed conic subset  $SS_k(F)$  of  $T^*X$  is defined by  $p \notin SS_k(F)$  if and only if the following condition is satisfied.

(i) There exists an open conic neighborhood U of p such that for any  $x \in \pi(U)$  and for any  $\mathbb{R}$ -valued C<sup>1</sup>-function  $\varphi$  defined on a neighborhood of x such that  $\varphi(x) = 0$ ,  $d\varphi(x) \in U$ , one has

(2.1) 
$$H^j_{\{\varphi \ge 0\}}(F)_x = 0 \quad \text{for any } j \le k.$$

We refer to [2] for equivalent definitions. In particular, it is proved in *loc. cit.* that one can replace the condition that  $\varphi$  is of class  $C^1$  by  $\varphi$  is of class  $C^{\alpha}$ , where  $\alpha \in \mathbb{Z}_{\geq 1} \cup \{\infty, \omega\}$ .

This condition is also equivalent to:

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(ii) there exists  $F' \in D^{\geq k+1}(\mathbf{k}_X)$  and an isomorphism  $F \simeq F'$  in  $D^b(\mathbf{k}_X; p)$ .

## 3. Normal cone

Let us recall the notion of the normal vector due to J.-M. Bony. Here we set for  $r \geq 0$ 

$$B_r(x) = \{ y \in \mathbb{R}^n ; \| y - x \| < r \},\$$

the open ball with center x and radius r.

PROPOSITION 3.1. — Let X be a real manifold, S a closed subset of X, and  $\alpha \in \mathbb{Z}_{\geq 1} \cup \{\infty, \omega\}$ . Then the following subsets of  $T^*X$  are equal:

- (i)  $SS_0(k_S);$
- (ii)<sub> $\alpha$ </sub> the closure of the set of points  $p = (x; \xi) \in T^*X$  such that  $x \in S$  and there is a  $C^{\alpha}$  function  $\varphi$  defined on a neighborhood U of x such that  $\varphi(x) = 0$ ,  $d\varphi(x) = \xi$  and  $S \cap U \subset \varphi^{-1}(\mathbb{R}_{\geq 0})$ ;
- (iii) the closure of the set of points  $p = (x;\xi) \in T^*X$  such that  $x \in S$  and  $C_x(S) \subset \{v \in T_xX; \langle v, \xi \rangle \ge 0\}.$

If X is an open subset of  $\mathbb{R}^n$ , then the above sets are equal to the following set:

(iv) the closure of the set of points  $p = (x; \xi) \in T^*X$  such that  $x \in S$  and the open ball  $B_{t \parallel \xi \parallel}(x - t\xi)$  does not intersect S for some t > 0.

*Proof.* — We may assume that  $X = \mathbb{R}^n$ .

 $(i) = (ii)_{\alpha}$  is an immediate consequence of Definition 2.2. In particular, the set  $(ii)_{\alpha}$  does not depend on  $\alpha$ .

- $(ii)_{\alpha} = (iv)$  for  $\alpha \ge 2$  is obvious.
- $(ii)_{\alpha} \subset (iii)$  is clear.
- (iii)  $\subset$  (iv). Let  $p = (x_0; \xi_0) \in T^*X$  with  $x_0 \in S$  and

$$C_{x_0}(S) \subset \{ v \in T_x X; \langle v, \xi_0 \rangle \ge 0 \}.$$

We have to prove that p belongs to the set (iv). The proof is very similar to that of Lemma 3.3 of [2], but for the sake of completeness, we shall repeat it.

If  $\xi_0 = 0$ , then it is trivially true. Hence we may assume that  $\xi_0 \neq 0$ .

We shall show that for an arbitrary open conic neighborhood U of p, there exists a point  $(x_1; \xi_1) \in U$  such that the open ball  $B_{\|\xi_1\|}(x_1 - \xi_1)$  satisfies

(3.1) 
$$x_1 \in S, \ x_1 \notin B_{\|\xi_1\|}(x_1 - \xi_1) \cap S.$$

This implies  $B_{t \parallel \xi_1 \parallel}(x_1 - t\xi_1) \cap S = \emptyset$  for  $0 < t \ll 1$ .

Let us take an open neighborhood V and a proper closed convex cone  $\gamma$  such that  $\xi_0 \in \text{Int}(\gamma^\circ), \overline{V} \times (\gamma^\circ \setminus \{0\}) \subset U$  and  $\text{Int}(\gamma) \neq \emptyset$ . Since  $C_{x_0}(S) \cap \gamma^a \subset \{0\}$ , there is  $\rho > 0$  such that

$$H_{-} := \left\{ x \in \mathbb{R}^{n} ; \left\langle x - x_{0}, \xi_{0} \right\rangle > -\rho \right\}$$

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satisfies  $\overline{H}_{-} \cap (x_0 + \gamma^a) \subset V$  and  $S \cap \overline{H}_{-} \cap (x_0 + \gamma^a) = \{x_0\}$ . Set  $S_0 := S \cap \overline{H}_{-}$ . Let us define the function  $\psi(x)$  on  $\mathbb{R}^n$  by

$$\psi(x) = \operatorname{dist}(x, \gamma^a) := \inf\{\|y - x\|; y \in \gamma^a\}.$$

It is well known that  $\psi$  is a continuous function on  $\mathbb{R}^n$ , and  $C^1$  on  $\mathbb{R}^n \setminus \gamma^a$ . More precisely for any  $x \in \mathbb{R}^n \setminus \gamma^a$ , there exists a unique  $y \in \gamma^a$  such that  $\psi(x) = ||x - y||$ . Moreover  $d\psi(x) = ||x - y||^{-1}(x - y) \in \gamma^\circ \setminus \{0\}$ . Furthermore  $B_{\psi(x)}(y)$  is contained in  $\{z \in \mathbb{R}^n; \psi(z) < \psi(x)\}$ .

For  $\varepsilon > 0$ , we set

$$\gamma^a_{\varepsilon} = \left\{ x \in \mathbb{R}^n \, ; \, \psi(x) < \varepsilon \right\}.$$

Then  $\gamma^a_{\varepsilon}$  is an open convex set. Moreover  $\gamma^a_{\varepsilon} + \gamma^a = \gamma^a_{\varepsilon}$ .

Let us take  $v \in \operatorname{Int}(\gamma)$ , and  $\delta > 0$  such that  $x_0 - \delta v \in H_-$ . Then take  $\varepsilon > 0$ such that  $\gamma_{\varepsilon}^a - \delta v \subset \operatorname{Int}(\gamma^a)$ ,  $(x_0 + \overline{\gamma_{\varepsilon}^a}) \cap \overline{H}_- \subset V$  and  $(x_0 + \overline{\gamma_{\varepsilon}^a}) \cap S_0 \subset H_-$ . Set  $W_t = x_0 + \gamma_{\varepsilon}^a + tv$  for  $t \in \mathbb{R}$ . Then one has

(3.2) 
$$W_t = \bigcup_{t' < t} W_{t'} \text{ and } \overline{W}_t = \bigcap_{t' > t} W_{t'} = \bigcap_{t' > t} \overline{W}_{t'},$$

(3.3)  $x_0 \in W_t \cap S \text{ for } t \ge 0, \text{ and } W_t \cap \overline{H}_- = \emptyset \text{ for } t \ll 0,$ 

(3.4) 
$$\overline{W}_t \cap S_0 \subset H_- \cap V \text{ for } t \leq 0.$$

Hence, for any closed subset K of  $\overline{H}_{-}$  and  $t \in \mathbb{R}$  such that  $K \cap \overline{W}_{t} = \emptyset$ , there exists t' > t such that  $K \cap W_{t'} = \emptyset$ .

Let us set  $c = \sup\{t; W_t \cap S_0 = \emptyset\}$ . Then  $c \leq 0$  and  $W_c \cap S_0 = \emptyset$ . By the above remark, one has  $\overline{W_c} \cap S_0 \neq \emptyset$ . Take  $x_1 \in S_0 \cap \partial W_c$ . Here  $\partial W_c := \overline{W_c} \setminus W_c$  is the boundary of  $W_c$ . The point  $x_1$  belongs to  $H_- \cap V$ .

As seen before, there exists an open ball  $B_{\varepsilon}(y)$  such that  $B_{\varepsilon}(y) \subset W_c$ ,  $||x_1 - y|| = \varepsilon$  and  $x_1 - y \in \gamma^{\circ}$ . Hence  $B_{\varepsilon}(y) \cap S \cap H_- = \emptyset$ , while  $H_-$  is a neighborhood of  $x_1$ . Hence  $(x_1; x_1 - y)$  belongs to U and satisfies condition (3.1).

DEFINITION 3.2. — For a closed subset S of X, the closed subset given in Proposition 3.1 is called the 0-conormal cone of S and denoted by  $N_0^*(S)$ .

Note that  $N_0^*(S)$  is a closed cone and it satisfies

$$\pi_X(N_0^*(S)) = \pi_X(N_0^*(S) \cap T_X^*X) = S.$$

Note also that one has  $N_0^*(S) = SS_0(\mathbf{k}_S) \subset SS(\mathbf{k}_S)$ .

EXAMPLE 3.3. — (i) If S = X, then  $N_0^*(S) = T_X^*X$ .

- (ii) If S is a closed submanifold, then  $N_0^*(S) = T_S^*X$ .
- (iii) If  $X = \mathbb{R}^2$  and  $S = \{(x, y) \in X ; x \ge 0 \text{ or } y \ge 0\}$ . Then

$$N_0^*(S) = \{(x, y; 0, 0) \in T_X^* X ; (x, y) \in S\}$$
  

$$\cup \{(x, y; \xi, \eta) \in T^* X ; y = 0, x \le 0, \eta \ge 0 \text{ and } \xi = 0\}$$
  

$$\cup \{(x, y; \xi, \eta) \in T^* X ; x = 0, y \le 0, \xi \ge 0 \text{ and } \eta = 0\}.$$

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