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HYPERBOLIC SYSTEMS ON NILPOTENT COVERS

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ABSTRACT. — We study the ergodicity of the weak and strong stable foliations of hyperbolic systems on nilpotent covers. Subshifts of finite type and geodesic flows on negatively curved manifolds are also considered.

RÉSUMÉ (*Systèmes hyperboliques sur des revêtements nilpotents*)
Nous étudions les propriétés ergodiques des feuilletages stables forts et faibles des systèmes hyperboliques définis sur un revêtement nilpotent. Les chaînes de Markov et les flots géodésiques en courbure négative sont aussi étudiés.

1. Introduction

This article is devoted to the study of the ergodicity of the stable foliation of an hyperbolic flow defined on a nilpotent cover of a manifold.

The geodesic flow on a compact two dimensional manifold of constant negative curvature is one of the simplest example of hyperbolic flow. E. Hopf [19], G.A. Hedlund [17], [18] showed the ergodicity of the associated horospheric flow with respect to the Liouville measure. H. Furstenberg obtained its unique ergodicity [15]. This was generalized to the case of the stable foliation of an

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hyperbolic flow by D.V. Anosov [2] and R. Bowen and B. Marcus [6]. In the context of infinite measure ergodic theory, it was natural to ask whether ergodicity of the foliation is again true on a regular cover.

This question has been investigated by many authors in the last decade. M. Babillot, F. Ledrappier studied the case of the geodesic flow on a compact manifold of constant negative curvature [3]. The extension of the flow to an abelian cover admits a number of invariant measures; they were able to show that the stable foliation of the extension is ergodic with respect to these measures. Their proof relies on counting estimates obtained by transfer operator technics. Using results from harmonic analysis, V. Kaimanovich was able to generalize their result to nilpotent extensions for the Liouville measure [20]. In another direction, M. Pollicott considered homologically full Anosov flows and obtained ergodicity of the foliation on an abelian cover for the measure of maximal entropy [23]. The method makes use of a symbolic model. We studied the case of Axiom A flows [10]; for these systems, the stable foliation may be non ergodic and we gave necessary and sufficient conditions in term of periodic orbits in order to get that ergodicity. Very few hypothesis were made on the measure, in order to treat the case of the Liouville measure, measure of maximal entropy and harmonic measure. Following M. Pollicott, we were working in a symbolic setting, so that two restrictive hypothesis were needed: the non-wandering set of the flow on the basis of the cover had to be compact and the cover was abelian. R. Solomyak [27], U. Hamenstadt [16] and then J. Aaronson, R. Solomyak and O. Sarig [1], also obtained several results, using methods of independent interest.

The goal of this article is to provide an unified approach to these problems and to recover these results using just the hyperbolicity of the systems. No compactness assumption on the wandering set is made and the cover can be taken nilpotent. As a consequence, we obtain a number of theorems, conjectured from the beginning, but which resisted all previous attempts, like the finite volume case.

We first state a general result concerning the ergodicity of the stable foliation of systems admitting local product structures. The second part of the article is devoted to the proof of that result. It is then applied to hyperbolic flows, subshifts of finite type and geodesic flows on negatively curved manifolds.

2. Main result

Let G be a Polish group acting continuously on a Polish space $\widehat{X}$ and $\widehat{\phi}_t$ a continuous flow on $\widehat{X}$ commuting with the G action. The quotient of $\widehat{X}$ by G is denoted by X, the projection from $\widehat{X}$ to X by π, and the quotient flow on X by ϕ_t. Let us assume that the quotient topology on X is given by a metric for which π is Lipschitz. Strong stable sets are defined by:

$W^{ss}(x) := \{y \in X \mid \lim_{t\to\infty} d(\phi_t(x), \phi_t(y)) = 0\}$;
$W^{ss}_\varepsilon(x) := \{y \in W^{ss}(x) \mid d(\phi_t(x), \phi_t(y)) \le \varepsilon \text{ for all } t \ge 0\}$;
$\widehat{W}^{ss}(\widehat{x}) := \{\widehat{y} \in \widehat{X} \mid \lim_{t\to\infty} d(\widehat{\phi}_t(\widehat{x}), \widehat{\phi}_t(\widehat{y})) = 0\}$;
$\widehat{W}^{ss}_\varepsilon(\widehat{x}) := \{\widehat{y} \in \widehat{W}^{ss}(\widehat{x}) \mid d(\widehat{\phi}_t(\widehat{x}), \widehat{\phi}_t(\widehat{y})) \le \varepsilon \text{ for all } t \ge 0\}$.

The weak stable sets $\widehat{W}^{ws}$, W^{ws} of $\widehat{\phi}_t$ and ϕ_t are equal to $\widehat{\phi}_{\mathbb{R}}(\widehat{W}^{ss})$ and $\phi_{\mathbb{R}}(W^{ss})$. One also defines the strong unstable sets $\widehat{W}^{su}$, $\widehat{W}^{su}_\varepsilon$, W^{su} and W^{su}_ε of $\widehat{\phi}_t$ and ϕ_t; these are the stable sets of $\widehat{\phi}_{-t}$ and ϕ_{-t}.

Let τ be a periodic orbit of ϕ_t on X and $\widehat{x} \in \widehat{X}$ a preimage of a point on the orbit τ. The period of τ is denoted by $\ell(\tau)$. The *Frobenius element* $[\tau]$ in G is the translation in the fiber along τ: $\widehat{\phi}_{\ell(\tau)}(\widehat{x}) = [\tau](\widehat{x})$. Its conjugacy class does not depend on the chosen preimage. If it belongs to the center of G, it is independant of the chosen preimage.

The next three uniformity assumptions are satisfied, for example, if the G-action on $\widehat{X}$ is isometric and the distance on X is locally equivalent to the distance $d'(x,y) = d_{\widehat{X}}(\pi^{-1}x, \pi^{-1}y)$.

• For all $g \in G$, the function $\widehat{x} \mapsto g\widehat{x}$ is uniformly continuous on $\widehat{X}$. In particular, this implies the relation $g\,\widehat{W}^{ss}(\widehat{x}) = \widehat{W}^{ss}(g\widehat{x})$ for all $\widehat{x}$.

• For all $v \in X$, $\varepsilon > 0$ and $T \ge 0$, there is a $\delta > 0$ such that for all $\widehat{v} \in \pi^{-1}(v)$, $d(\widehat{x}, \widehat{v}) < \delta$ implies $d(\widehat{\phi}_t(\widehat{x}), \widehat{\phi}_t(\widehat{v})) < \varepsilon$ for all $|t| \le T$.

• Let v some point in X. If $\{\widehat{x}_n\}$ is a sequence in $\widehat{X}$, $d(\pi(\widehat{x}_n), v) \to 0$ implies $d(\widehat{x}_n, \pi^{-1}v) \to 0$.

The strong unstable foliation $\widehat{W}^{su}$ is said to be *locally contracting uniformly in fibers* if for any $v \in X$, there is an $\varepsilon > 0$ such that for any $\delta_0 > 0$, one can find a $t_0 \in \mathbb{R}$ so that $d(\widehat{\phi}_{-t}(\widehat{x}), \widehat{\phi}_{-t}(\widehat{y})) \le \delta_0$ whenever $t \ge t_0$, $d(\widehat{x}, \pi^{-1}v) < \varepsilon$, $d(\widehat{y}, \pi^{-1}v) < \varepsilon$ and $y \in \widehat{W}^{su}_\varepsilon(x)$.

The flow $\widehat{\phi}_t$ is said to admit a *local product structure* if, for all points $v \in X$, and all $\varepsilon > 0$, there exists an $\varepsilon_0 > 0$, a neighborhood V of $\pi^{-1}v$ with $\varepsilon_0 < d(\pi^{-1}v, V^c)$, $d(x, \pi^{-1}v) < \varepsilon$ for all $x \in V$, and positive constants δ_1, δ_2 less than ε, such that for all $x, y \in V$ with $d(x,y) \le \delta_1$, there is a point $\langle x, y\rangle \in V$, a real number t with $|t| \le \delta_2$, and an element $g \in G$ with $d(g, \mathrm{id}) \le \delta_2$, so that:

$$\widehat{W}^{ss}_{\delta_2}\big(g\widehat{\phi}_t(x)\big) \cap \widehat{W}^{su}_{\delta_2}(y) = \langle x, y\rangle.$$

The maps sending (x,y) to $\langle x, y\rangle$, g and t are furthermore supposed to be Borel maps; in practice, they will even be continuous on V.

The spaces $\widehat{X}$ and X are endowed with two Borel measures, $\widehat{\mu}$ and μ, in such a way that the projection $\pi : \widehat{X} \to X$ is non-singular: inverse images of sets of zero measure are of zero measure. These measures are supposed to be invariant by the flows $\widehat{\phi}_t$ and ϕ_t; the space $\widehat{X}$ admits a countable cover by open sets of finite $\widehat{\mu}$-measure, whereas μ is a probability measure ergodic with respect to the flow ϕ_t. The measure $\widehat{\mu}$ satisfies the following conditions: there is a neighborhood of the identity in G and a constant C such that $g_*\widehat{\mu} \leq C\widehat{\mu}$ for g in the neighborhood. We also suppose:

Given a periodic vector v in X, let V be a neighborhood of $\pi^{-1}v$ coming from the local product structure. Consider the set A of points $\widehat{x}$ in V, for which $[v]^{-1}\widehat{\phi}_{\ell(v)}(\widehat{x})$ is again in V and δ_1-close to $\widehat{x}$, and the transformation T on A which associate to the point $\widehat{x}$ in A the point $\langle [v]^{-1}\widehat{\phi}_{\ell(v)}(\widehat{x}), \widehat{x}\rangle$. Then we suppose that there is a constant C depending only on v, such that $\widehat{\mu}(T^{-1}B \cap A) \leq C\widehat{\mu}(B)$ for all Borel set B in $\widehat{X}$.

This condition may seem awkward; in practice, it follows easily from the absolute continuity of the measure with respect to the holonomy of the foliations. However, since no regularity on the foliations is assumed, one has to give a more technical statement.

Finally, recall that the partition of $\widehat{X}$ by the stable sets $\widehat{W}^{ss}$ is said to be ergodic if any union of elements of the partition that forms a Borel set is of measure 0 or with complement of measure 0. In the following, this partition will be called abusively a foliation, and the sets $\widehat{W}^{ss}$ will be called the leaves of the foliation.

We can now state the main result:

THEOREM 1. — *Let G be a Polish group acting continuously on a Polish space $\widehat{X}$ and $\widehat{\phi}_t$ a continuous flow on $\widehat{X}$ commuting with the G action. The system satisfies the previous hypothesis: three uniformity conditions, local contraction of $\widehat{W}^{su}$ uniformly in fibers, and local product structure. Moreover X and $\widehat{X}$ are equipped with measures μ and $\widehat{\mu}$ as defined above.*

Suppose that G is a nilpotent group admitting a central series composed of closed subgroups $\{1\} = G_0 \subset G_1 \subset \cdots \subset G_n = G$ so that, for all $i \geq 1$, the subgroup of G_i/G_{i-1} generated by the Frobenius elements in G_i of periodic orbits of ϕ_t is dense in G_i/G_{i-1}:

$$\overline{\big\langle \{[\tau]\,;\ \tau \text{ periodic orbit of } \phi_t \text{ with } [\tau] \in G_i\}\big\rangle} = G_i/G_{i-1},$$

then the weak stable foliation $\widehat{W}^{ws}$ is ergodic with respect to $\widehat{\mu}$.

If moreover the following is satisfied:

$$\{0\} \times G_i/G_{i-1} \subset \overline{\big\langle \{(\ell(\tau), [\tau])\,;\ \tau \text{ periodic orbit of } \phi_t \text{ with } [\tau] \in G_i\}\big\rangle},$$

$$\overline{\big\langle \{\ell(\tau)\,;\ \tau \text{ periodic orbit of } \phi_t\}\big\rangle} = \mathbb{R},$$

then the strong stable foliation $\widehat{W}^{ss}$ is ergodic with respect to $\widehat{\mu}$.

REMARKS. — • The periodic orbits of ϕ_t with Frobenius elements in G_i are precisely the orbits whose lifts in $\widehat{X}/G_i$ are again periodic.

• The conditions for the ergodicity of the strong stable foliation can be rewritten as $\overline{\langle\{[\tau]' ; [\tau]' \in G_i'\}\rangle} = G_i'/G_{i-1}'$ for all $i \geq 1$, with the convention $G' = \mathbb{R} \times G$, $G_i' = \{0\} \times G_i'$ if $i < n$, and $[\tau]' = (\ell(\tau), [\tau])$.

• There are systems satisfying the equalities $\{[\tau] ; [\tau] \in G_i\} = G_i/G_{i-1}$ and $\overline{\langle\{\ell(\tau) ; [\tau] \in G_i\}\rangle} = \mathbb{R}$, for all $i \geq 1$, and whose strong stable foliation is not ergodic. An example is provided in the section dealing with subshifts of finite type. In that example, the flow $\widehat{\phi}_t$ has no periodic orbits on $\widehat{X}$.

• If the subgroup generated by the lengths of the periodic orbits of $\widehat{\phi}_t$ on $\widehat{X}$ is dense in $\mathbb{R}$, then the conditions $\overline{\langle\{[\tau] ; [\tau] \in G_i\}\rangle} = G_i/G_{i-1}$, for all i, are easily seen to imply the ergodicity of both foliations on $\widehat{X}$.

A similar result can be proven for transformations instead of flows; here are the modifications to introduce in that case: the group $\mathbb{R}$ should be replaced by $\mathbb{Z}$ whenever it occurs; in the definition of the local product structure, the constant δ_2 is removed and $t(x, y)$ is equal to zero. The statement of the theorem now translates verbatim, and there is essentially no change in the proof.

3. Proof of the ergodicity

This section is devoted to the proof of the main theorem. The central series of G gives a sequence of systems:

$$\widehat{X} \to \widehat{X}/G_1 \to \cdots \to \widehat{X}/G_{i-1} \to \widehat{X}/G_i \to \cdots \to \widehat{X}/G_{n-1} \to \widehat{X}/G = X.$$

Ergodicity of the strong stable foliation is proven by a recurrence on this series. Assuming that the strong stable foliation of the quotient flow on $\widehat{X}/G_i$ is ergodic, one has to show that it is again ergodic on $\widehat{X}/G_{i-1}$. This amounts to showing that Borel functions $F : \widehat{X}/G_{i-1} \to \mathbb{R}$ invariant by the strong stable foliation of the quotient flow on $\widehat{X}/G_{i-1}$ are invariant by the action of G_i/G_{i-1}; indeed this implies that F factorizes through an invariant function defined on $\widehat{X}/G_i$, and is thus constant by the recurrence hypothesis.

The following set is easily seen to be a closed subgroup of $\mathbb{R} \times G/G_{i-1}$:

$$\big\{(t, g) ; \textit{for all } \widehat{W}^{ss} \textit{invariant } F \textit{ in } L^\infty(\widehat{X}/G_{i-1}),\ F(\widehat{\phi}_{-t} gx) = F(x)\big\}$$

The invariant functions will factorize through $\widehat{X}/G_i$ if $\{0\} \times G_i/G_{i-1}$ is included in this set. In order to obtain this inclusion, it will be shown that the lengths and the Frobenius elements $(\ell(\tau), [\tau])$ of the periodic orbits τ of ϕ_t are included in this set, if $[\tau]$ belongs to G_i.

LEMMA 1. — *The closed group*

$$\big\{(t, g) \in \mathbb{R} \times G/G_{i-1} ; \forall F,\ \widehat{W}^{ss} \textit{invariant on } \widehat{X}/G_{i-1},\ F(\widehat{\phi}_{-t} gx) = F(x)\big\}$$