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EQUIDISTRIBUTION TOWARDS THE GREEN CURRENT

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ABSTRACT. — Let $f : \mathbb{P}^k \to \mathbb{P}^k$ be a dominating rational mapping of first algebraic degree $\lambda \geq 2$. If S is a positive closed current of bidegree $(1,1)$ on $\mathbb{P}^k$ with zero Lelong numbers, we show – under a natural dynamical assumption – that the pullbacks $\lambda^{-n}(f^n)^*S$ converge to the Green current T_f. For some families of mappings, we get finer convergence results which allow us to characterize all f^*-invariant currents.

RÉSUMÉ (*Équidistribution vers le courant de Green*). — Soit $f : \mathbb{P}^k \to \mathbb{P}^k$ une application rationnelle dominante de premier degré algébrique $\lambda \geq 2$. Lorsque S est un courant positif fermé de bidegré $(1,1)$ sur $\mathbb{P}^k$ dont les nombres de Lelong sont tous nuls, nous montrons, sous une hypothèse dynamique naturelle, que les pull-backs $\lambda^{-n}(f^n)^*S$ convergent vers le courant de Green T_f. Pour certaines familles d'applications, des résultats de convergence raffinés nous permettent de caractériser tous les courants f^*-invariants.

Introduction

Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be a rational map of degree $\lambda \geq 2$. A celebrated result of Brolin, Lyubich, Freire-Lopez-Mañe asserts that the preimages $\lambda^{-n}(f^n)^*\sigma$ of any probability measure σ on $\mathbb{P}^1$ converge to an invariant measure μ_f as soon as $\sigma(\mathcal{E}_f) = 0$, where $\mathcal{E}_f$ is a (possibly empty) finite exceptional set. The purpose of this note is to prove similar results in higher dimension.

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Let $f : \mathbb{P}^k \to \mathbb{P}^k$ be a rational mapping. It can be written $f = [P_0 : \cdots : P_k]$ in homogeneous coordinates, where the P_j's are homogeneous polynomials of the same degree λ (the first algebraic degree of f) with no common factor $P_0 \wedge \cdots \wedge P_k = 1$. Note that when $k \geq 2$, f is not necessarily holomorphic: it is not well defined on the indeterminacy set $I_f = \bigcap_j P_j^{-1}(0)$ which is an algebraic subset of $\mathbb{P}^k$ of codimension ≥ 2. There are several ways one can try to generalize the one-dimensional result. Given Z an algebraic subset of $\mathbb{P}^k$ of pure codimension p, one can ask whether $f^{-n}(Z)$ (properly normalized) converges to an invariant current of bidegree (p, p). In this note we focus on the case $p = 1$.

Given S a positive closed current of bidegree $(1, 1)$ on $\mathbb{P}^k$, we consider

$$S_n := \lambda^{-n}(f^n)^* S.$$

This is a bounded sequence of positive closed currents of bidegree $(1, 1)$ on $\mathbb{P}^k$. When $S = \omega$ is the Fubini-Study Kähler form, it was proved by Sibony [19] that (ω_n) converges to an invariant Green current T_f. On the other hand Russakovskii and Shiffman have shown [18] that $[H]_n - \omega_n \to 0$ for almost every hyperplane H of $\mathbb{P}^k$. Our main result interpolates between these two extreme cases.

THEOREM 0.1. — *Let $f : \mathbb{P}^k \to \mathbb{P}^k$ be a dominating rational mapping with $\lambda \geq 2$. Assume there exists an invariant probability measure μ such that $\log |J_{FS}(f)| \in L^1(\mu)$. Let S be a positive closed current of bidegree $(1, 1)$ and unit mass on $\mathbb{P}^k$. If $\nu(S, p) = 0$ for all $p \in \mathbb{P}^k$, then*

$$\frac{1}{\lambda^n}(f^n)^* S \longrightarrow T_f \quad \textit{in the weak sense of currents.}$$

Here $\nu(S, p)$ denotes the Lelong number of S at point p and $J_{FS}(f)$ denotes the jacobian of f with respect to the Fubini-Study volume form ω^k. Similar (weaker) results were previously obtained for Hénon mappings [1], [8], birational mappings [6], and holomorphic endomorphisms of $\mathbb{P}^k$ [9], [19], [7].

Although Theorem 0.1 does not imply directly Russakovskii-Shiffman's result, the proof shows one essentially has to assume $\sup_{p \in \mathbb{P}^k \setminus E} \nu(S_n, p) \to 0$, where E is some (possibly empty) exceptional set (see Theorem 1.4). The key ingredients of the proof are: a pluripotential estimate of the volume of sublevel sets of a quasiplurisubharmonic function [16] and a dynamical estimate on the decreasing of volumes under iteration (Theorem 1.2). Note that all the volumes are computed with respect to the Fubini-Study volume form ω^k.

We prove the volume estimates and our main result in Section 1. We give refinements of the latter in Section 2 in case f is an holomorphic endomorphism of $\mathbb{P}^k$ (Section 2.1) or a special type of polynomial endomorphism of $\mathbb{C}^k$ (Section 2.2). This allows us to characterize every f^*-invariant current. Such equidistribution results should be understood as strong ergodic properties of the Green current T_f. In dimension 1 indeed this implies that T_f is strongly mixing (see Theorem VIII.22 in [2]). For the reader's convenience we recall in an Appendix compactness criteria for families of qpsh functions. They are the higher dimensional counterparts of Montel's Theorem.

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1. Equidistribution of pullbacks of currents

Let $f : \mathbb{P}^k \to \mathbb{P}^k$ be a rational mapping with first algebraic degree $\lambda \geq 2$. We always assume f is dominating, *i.e.* its jacobian does not vanish identically in any coordinate chart. It follows that a generic point has $d_t(f)$ well defined preimages by f. Note that $d_t(f) = \lambda$ when $k = 1$ but these two degrees differ in general when $k \geq 2$.

Let ω denote the Fubini-Study Kähler form on $\mathbb{P}^k$. The smooth form $f^*\omega$ is well defined in $\mathbb{P}^k \setminus I_f$ and extends trivially through I_f as a positive closed current of bidegree $(1,1)$ and mass $\|f^*\omega\| = \int_{\mathbb{P}^k} f^*\omega \wedge \omega^{k-1} = \lambda$. So $\lambda^{-1} f^*\omega$ is cohomologous to ω. Since $\mathbb{P}^k$ is Kähler, this can be written

$$\lambda^{-1} f^*\omega = \omega + \mathrm{dd}^c G,$$

where $G \in L^1(\mathbb{P}^k)$ (see [11, p. 149]). The function G is "quasiplurisubharmonic" (qpsh): it is locally given as the sum of a psh function (a local potential of $\lambda^{-1} f^*\omega \geq 0$) and a smooth function (a local potential of $-\omega$). In particular it is bounded from above on $\mathbb{P}^k$: replacing G by $G - C_0$, we can therefore assume $G \leq 0$. Sibony [19] has shown that the decreasing sequence of qpsh functions

$$(*) \qquad G_n := \sum_{j=0}^{n-1} \frac{1}{\lambda^j} G \circ f^j$$

converges in $L^1(\mathbb{P}^k)$ to a qpsh function $G_\infty \in L^1(\mathbb{P}^k)$. This shows that $\lambda^{-n}(f^n)^*\omega$ converges in the weak sense of (positive) currents to the so called Green current $T_f \geq 0$ which satisfies $f^*T_f = \lambda T_f$.

A natural question is then to look at the convergence of $S_n := \lambda^{-n}(f^n)^*S$, where S is now any positive closed current of bidegree $(1,1)$ and unit mass on $\mathbb{P}^k$. When $S = [H]$ is the current of integration along an hyperplane of $\mathbb{P}^k$, it was shown by Russakovskii and Shiffman [18] that $[H]_n \to T_f$ for every H outside some pluripolar set $\mathcal{E} \subset (\mathbb{P}^k)^*$. In order to prove convergence of S_n for more general currents S, we first need to get control on the decreasing of volumes under iteration.

1.1. Volume estimates. — Let $f : \mathbb{P}^k \to \mathbb{P}^k$ be a dominating rational mapping with $\lambda \geq 2$. Let $J_{FS}(f)$ denote its jacobian with respect to the Fubini-Study Kähler volume form. It is defined by

$$f^*\omega^k = \big|J_{FS}(f)\big|^2\omega^k.$$

PROPOSITION 1.1. — *Fix B an open subset of $\mathbb{P}^k$ and $\delta_0 > 0$. There exists $C_0 > 0$ such that for every open subset Ω of $\mathbb{P}^k$ with* $\mathrm{vol}(\Omega) \geq \delta_0$*,*

$$\mathrm{vol}\big(f^n(\Omega)\big) \geq (C_0)^{\lambda^n} \exp\Big(\frac{1}{\mathrm{vol}(B)\,\mathrm{vol}(\Omega)} \int_B \log\big|J_{FS}(f^n)\big|^2\omega^k\Big)$$

for all $n \in \mathbb{N}$.

Proof. — Fix $\mathbb{C}^k$ an affine chart of $\mathbb{P}^k$. We have

$$f = (P_1/P_0, \ldots, P_k/P_0)$$

in $\mathbb{C}^k$ where the P_j's are polynomials of degree $\leq \lambda = \delta_1(f)$. Since $\omega = \mathrm{dd}^c\frac{1}{2}\log[1 + \|z\|^2]$ in $\mathbb{C}^k$, we get

$$\omega^k(z) = \big(1 + \|z\|^2\big)^{-(k+1)}\mathrm{d}V,$$

where $\mathrm{d}V$ denotes the euclidean volume form in $\mathbb{C}^k$. Therefore

$$\big|J_{FS}(f)\big|^2 = \big|J_{\mathrm{eucl}}(f)\big|^2\Big(\frac{1 + \|z\|^2}{1 + \|f(z)\|^2}\Big)^{k+1}.$$

We infer

$$\log|J_{FS}(f)| = u - v,$$

where u, v are qpsh functions such that $\mathrm{dd}^c u, \mathrm{dd}^c v \geq -2\lambda k\omega$. Let Ω be an open subset of $\mathbb{P}^k$. We have

$$\mathrm{vol}\big(f^n(\Omega)\big) = \int_{f^n(\Omega)} \omega^k \geq \frac{1}{d_t(f)^n}\int_\Omega \big|J_{FS}(f^n)\big|^2\omega^k,$$

where the inequality follows from the change of variable formula. The concavity of the logarithm yields

$$\mathrm{vol}\big(f^n(\Omega)\big) \geq \frac{\mathrm{vol}(\Omega)}{d_t(f)^n}\exp\Big[\frac{2D_n}{\mathrm{vol}(\Omega)}\int_\Omega \frac{1}{D_n}(u_n - v_n)\omega^k\Big],$$

where

$$\log|J_{FS}(f^n)| = u_n - v_n$$

with $\mathrm{dd}^c u_n, \mathrm{dd}^c v_n \geq -D_n\omega$, $D_n \leq 2\lambda^n k$. Observe that we can decompose

$$\frac{1}{D_n}\log\big|J_{FS}(f^n)\big| = \varphi_n - \psi_n + \frac{1}{\mathrm{vol}(B)}\int_B \frac{1}{D_n}\log\big|J_{FS}(f^n)\big|\omega^k,$$

where

$$\varphi_n = D_n^{-1}u_n - \log\|z\| - \frac{1}{\mathrm{vol}(B)}\int_B \big(D_n^{-1}u_n - \log\|z\|\big)\omega^k,$$
$$\psi_n = D_n^{-1}v_n - \log\|z\| - \frac{1}{\mathrm{vol}(B)}\int_B \big(D_n^{-1}v_n - \log\|z\|\big)\omega^k.$$

The functions φ_n, ψ_n are quasiplurisubharmonic on $\mathbb{P}^k$ ($\mathrm{dd}^c\varphi_n, \mathrm{dd}^c\psi_n \geq -\omega$) with $\int_B \varphi_n = \int_B \psi_n = 0$. It follows therefore from Proposition 3.2 (Appendix) that they belong to a compact family of qpsh functions, so there exists $C_\Omega \in \mathbb{R}$ such that $\int_\Omega(\varphi_n - \psi_n)\omega^k \geq C_\Omega$, for all $n \in \mathbb{N}$. Since $D_n \leq 2k\lambda^n$, this yields the desired inequality. □

It remains to get a lower bound on $\int_B \log|J_{FS}(f^n)|\omega^k$, where B is an open subset which we may fix as we like.

Theorem 1.2. — *Let $f : \mathbb{P}^k \to \mathbb{P}^k$ be a dominating rational mapping with $\lambda \geq 2$. Assume there exists an invariant probability measure μ such that $\log|J_{FS}(f)| \in L^1(\mu)$. Fix $\delta_0 > 0$. Then there exists $C_0 > 0$ such that for every open subset Ω of $\mathbb{P}^k$ with* $\mathrm{vol}(\Omega) \geq \delta_0$,

$$\mathrm{vol}\big(f^n(\Omega)\big) \geq C_0^{\lambda^n}, \quad \forall n \in \mathbb{N}.$$

Proof. — Using Proposition 1.1, it is sufficient to find $M > 0$ such that for all n, $\int_B \log|J_{FS}(f^n)|\omega^k \geq -M\lambda^n$. We take here $B = \mathbb{P}^k$ (but other normalisations could be useful, see Remark 1.3 below).

We decompose $\lambda^{-1}\log|J_{FS}(f)| = u - v + C$, where u, v are qpsh functions ($\mathrm{dd}^c u, \mathrm{dd}^c v \geq -\omega$) such that $\sup_{\mathbb{P}^k} u = \sup_{\mathbb{P}^k} v = 0$ and $C \in \mathbb{R}$. Thus we get

$$\frac{1}{\lambda^n}\log\big|J_{FS}(f^n)\big| = \frac{1}{\lambda^n}\sum_{j=0}^{n-1}\log\big|J_{FS}(f)\circ f^j\big| \geq \sum_{j=0}^{n-1}\frac{1}{\lambda^{n-j}}u_j + \frac{n}{\lambda^n}C,$$

where $u_j := \lambda^{-j}u \circ f^j$. It is therefore sufficient to get a uniform lower bound on $\int_{\mathbb{P}^k} u_j\omega^k$. This is a consequence of the fact that (u_j) is relatively compact in $L^1(\mathbb{P}^k)$. Indeed $\mathrm{dd}^c u_j \geq -\lambda^{-j}(f^j)^*\omega$, so $u_j + G_j$ is qpsh. By Lemma 3.1 (Appendix), the sequence $(u_j + G_j)$ is either relatively compact or uniformly converges to $-\infty$. Since $G_j \to G_\infty \in L^1(\mathbb{P}^k)$, the sequence (u_j) is either relatively compact or converges to $-\infty$. But the latter can not happen since $u \in L^1(\mu)$ and $\int u_j\,\mathrm{d}\mu = \lambda^{-j}\int u\,\mathrm{d}\mu \to 0$. The desired control on $\int_{\mathbb{P}^k}\log|J_{FS}(f^n)|\omega^k$ follows. □

Remark 1.3. — The assumption on the existence of μ is natural in our dynamical context. Observe that it is satisfied if *e.g.* there exists a non critical periodic point.

Other assumptions could be made to obtain the final lower bound on $\int_B \log|J_{FS}(f^n)|\omega^k$. If $f_{|\mathbb{C}^k}$ is polynomial, it is enough to assume that