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DIMENSION OF WEAKLY EXPANDING POINTS FOR QUADRATIC MAPS

by Samuel Senti

Abstract. — For the real quadratic map $P_a(x) = x^2 + a$ and a given $\epsilon > 0$ a point x has good expansion properties if any interval containing x also contains a neighborhood J of x with $P_a^n|_J$ univalent, with bounded distortion and $B(0,\epsilon) \subseteq P_a^n(J)$ for some $n \in \mathbb{N}$. The ϵ-weakly expanding set is the set of points which do not have good expansion properties. Let α denote the negative fixed point and M the first return time of the critical orbit to $[\alpha, -\alpha]$. We show there is a set $\mathcal{R}$ of parameters with positive Lebesgue measure for which the Hausdorff dimension of the ϵ-weakly expanding set is bounded above and below by $\log_2 M/M + \mathcal{O}(\log_2\log_2 M/M)$ for ϵ close to $|\alpha|$. For arbitrary $\epsilon \leq |\alpha|$ the dimension is of the order of $\mathcal{O}(\log_2|\log_2\epsilon|/|\log_2\epsilon|)$. Constants depend only on M. The Folklore Theorem then implies the existence of an absolutely continuous invariant probability measure for P_a with $a \in \mathcal{R}$ (Jakobson's Theorem).

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Résumé (*Dimension des points faiblement dilatants pour l'application quadratique*)

Pour l'application quadratique réelle $P_a(x) = x^2 + a$ et un $\epsilon > 0$ donné, un point x a de bonnes propriétés de dilatation si tout intervale contenant x contient également un voisinage J de x avec $P_a^n|_J$ univalent, avec distortion bornée et $B(0,\epsilon) \subseteq P_a^n(J)$ pour un $n \in \mathbb{N}$. L'ensemble ϵ-faiblement dilatant est l'ensemble des points qui n'ont pas de bonnes propriétes de dilatation. Notons α le point fixe négatif et M le temps de premier retour de l'orbite critique dans $[\alpha, -\alpha]$. Nous prouvons l'existence d'un ensemble $\mathcal{R}$ de paramètres de mesure de Lebesgue positive pour lesquels la dimension de Hausdorff de l'ensemble ϵ-faiblement dilatant est bornée supérieurement et inférieurement par $\log_2 M/M + \mathcal{O}(\log_2 \log_2 M/M)$ si ϵ est proche de $|\alpha|$. Pour $\epsilon \leq |\alpha|$ quelconque la dimension est de l'ordre de $\mathcal{O}(\log_2 |\log_2 \epsilon|/|\log_2 \epsilon|)$. Les constantes ne dependent que de M. Le théorème du Folklore implique alors l'existence d'une mesure de probabilité absolument continue et invariante par P_a pour $a \in \mathcal{R}$ (théorème de Jakobson).

1. Introduction

The 1-parameter family of real quadratic maps $P_a(x) = x^2 + a$ is a simple model of nonlinear dynamics exhibiting surprisingly rich dynamical structure and giving rise to interesting and difficult questions. For instance, the existence of a P_a-invariant probability measure which is absolutely continuous with respect to the Lebesgue measure *Leb*, a so-called a.c.i.p. , for a set of parameters of positive Lebesgue measure was first proved by Jakobson in [9] (see also [4], [14], [16], [10], [17], [15]).

Much is known about the real quadratic family, including a positive answer for the real Fatou conjecture (see [8], [11]) as well as for its generalization to real analytic families of maps with one non-flat critical point and negative Schwarzian derivative, so-called S-unimodal families [1]. Avila and Moreira have also shown that almost every non-regular parameter (no periodic hyperbolic attractor) satisfies the Collet-Eckmann condition for the quadratic map [3] as well as for a residual set of analytical S-unimodal families with a quadratic critical point [2]. However, many questions remain for S-unimodal maps, Hénon maps (see [5]) and complex quadratic maps. The real quadratic map enjoys a particular status as it serves as a model to help understand these problems.

A common technique to build an a.c.i.p. for an interval map f is to look for expansion for some iterate of the original map on a properly restricted domain J_i. The map $T|_{J_i} := f^{n_i}|_{J_i}$ is called the *induced map*. The Folklore Theorem (see for instance [12], [6], [13]) asserts the existence of a T-invariant a.c.i.p. ν provided there exists $\epsilon > 0$ and a T-invariant interval A containing the critical point satisfying the following three conditions: a) A is the countable union, modulo sets of zero Lebesgue measure, of intervals J_i with disjoint interiors, b) T restricted to each J_i is a diffeomorphism with uniformly bounded distortion and c) $Leb(T(J_i)) \geq \epsilon$. If additionally the summability condition

$\sum n_i \nu(J_i) < \infty$ holds, then there exists an a.c.i.p. μ which is invariant by the original map f.

Let $B(0,\epsilon)$ denote the ball of radius ϵ centered at 0. For the quadratic family P_a and a given $\epsilon > 0$, we will say that a point x has good expansion properties if any interval containing x also contains a neighborhood J of x with $P_a^n|_J$ univalent with bounded distortion and $B(0,\epsilon) \subseteq P_a^n(J)$ for some iterate $n \in \mathbb{N}$. We call points which do not have good expansion properties ϵ-*weakly expanding*. For such points there are only a finite number of iterates for which one can find a neighborhood J of x with $P_a^n|_J$ univalent with bounded distortion and $B(0,\epsilon) \subseteq P_a^n(J)$

In this work, we estimate the Hausdorff dimension of the ϵ-weakly expanding set for the quadratic map for a set of parameters of positive Lebesgue measure. More precisely if α is the negative fixed point and

$$M := \min\bigl\{n \in \mathbb{N}\,;|P_a^n(0)| < |\alpha|\bigr\}$$

is the first return time of the critical orbit to $[\alpha, -\alpha]$, we prove the following:

Main Theorem. — *For a set of parameters $\mathcal{R}$ with*

$$\lim_{\delta\to 0} \frac{Leb(\mathcal{R} \cap [-2, -2+\delta))}{\delta} = 1$$

the Hausdorff dimension of the ϵ-weakly expanding set $\mathcal{E}_\epsilon$ is

$$\dim_H(\mathcal{E}\epsilon) = \frac{\log_2 M}{M}\Bigl(1 + \mathcal{O}\Bigl(\frac{\log_2 \log_2 M}{\log_2 M}\Bigr)\Bigr)$$

for $c\,2^{-M} < \epsilon \le |\alpha|$ and a constant $c > 0$. For $0 < \epsilon \le |\alpha|$ arbitrary, we have

$$\dim_H(\mathcal{E}_\epsilon) = \mathcal{O}\Bigl(\frac{\log_2 |\log_2 \epsilon|}{|\log_2 \epsilon|}\Bigr).$$

Constants depend only on the return time M.

For parameters in $\mathcal{R}$ close to -2 the return time M is large, so points with good expansion properties have full Lebesgue measure. For $\epsilon = |\alpha|$ the Folklore Theorem applies and Jakobson's Theorem follows as a Corollary once we prove the summability condition.

Corollary (Jakobson [9]). — *For the real quadratic map P_a, there exists a probability measure which is invariant by P_a and absolutely continuous with respect to the Lebesgue measure for a set of parameters $a \in \mathcal{R}$ of positive Lebesgue measure.*

Structure of the paper. — We consider the neighborhood $A := [\alpha, -\alpha]$ of the critical point in Section 2. We build a partition of A consisting of those preimages of A on which the induced map T can be extended as a diffeomorphism to a given larger domain containing A. Such extendibility insures uniform bounded distortion by Koebe's distortion property and the image by T of the partition elements obviously contain the ball of radius $\epsilon < |\alpha|$ centered at 0.

In Section 3 we identify the sets that contain ϵ-weakly expanding points with $\epsilon = \alpha$.

In Section 4 we introduce Yoccoz's strongly regular parameter conditions [17] and show that for these parameters the α-weakly expanding set is the attractor of an iterated function system (IFS) with countably many generators.

In Section 5, we obtain derivative estimates for the generators of the IFS by viewing the quadratic map as a perturbation of the Chebyshev polynomial x^2-2 for parameters close to -2 (large M). We also prove that strongly regular parameters satisfy the Collet-Eckmann condition.

In Section 6 we bound the Hausdorff dimension of the α-weakly expanding set. The lower bound is obtained by approximating this set by the attractor of an IFS with finitely many generators and estimating the dimension of the approximation (which obviously contains the α-weakly expanding set). If the number of generators of the approximation is large enough, the difference between both dimensions lies within the error bounds produced by the derivative estimates giving us an upper bound. The complement of the partition of A coincides with the ϵ-weakly expanding set with $c2^{-M} < \epsilon \leq |\alpha|$ for some constant $c > 0$. We prove a measure estimate which implies both the summability condition and the full Lebesgue measure of the partition of A. The Folklore Theorem together with the positive Lebesgue measure of strongly regular parameters (see [17], [15]) then implies Jakobson's Theorem [9].

In Section 7 we consider the ϵ-weakly expanding set for general ϵ. We generalize the construction introduced in the first part, showing that the ϵ-weakly expanding set is the attractor of an IFS with countably many generators. In this case only very crude derivative estimates are necessary. We bound the number of contractions for which the same estimates are used and proceed similarly as in the first part to estimate the dimension of the ϵ-weakly expanding set.

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2. Definitions

For the real quadratic map

$$P_a(x) = x^2 + a$$

with parameters $-\frac{3}{4} > a \in \mathbb{R}$ denote the fixed points by $\beta > 0$ and $\alpha < 0$. For $n \in \mathbb{N}$, set $\alpha^0 := \alpha$ and define inductively α^n as the unique negative point for which

$$P_a(\alpha^n) = -\alpha^{n-1}.$$

If $a < \alpha^{n-1}$ define inductively $\widetilde{\alpha}^n$ as the unique negative point for which

$$P_a(\widetilde{\alpha}^n) = \alpha^{n-1}.$$

Choose $M \in \mathbb{N}$ large and consider parameters for which $P_a(0) \in (\alpha^{M-1}, \alpha^{M-2})$. Let

$$A := [\alpha, -\alpha] \subseteq [\alpha^1, -\alpha^1] =: \widehat{A}.$$

Set $C_1^+ = [\alpha^1, \alpha]$ and

$$C_n^+ := [\widetilde{\alpha}^{n-1}, \widetilde{\alpha}^n] \quad \text{for } 2 \le n \le M-2.$$

Also write $C_n^- = -C_n^+$ for the interval symmetric to C_n^+.

Definition 2.1. — An interval $J \subseteq [-\beta, \beta]$ is *regular of order* $n \geq 0$ if there exists a neighborhood $\widehat{J}$ of J such that the restriction of P_a^n to $\widehat{J}$ is a diffeomorphism onto $\widehat{A}$ and $P_a^n(J) = A$. We denote the order of a regular interval by *ord*(J). Denote

$$\mathcal{J} := \big\{J \subsetneq A \ \text{ regular}\,; \forall J' \text{ regular with } \operatorname{int}(J') \cap \operatorname{int}(J) \neq \varnothing \Rightarrow J' \subseteq J\big\}.$$

Elements of $\mathcal{J}$ are maximal with respect to inclusion.

Proposition 2.2. — a) *Regular intervals are nested or have disjoint interiors.*

b) $[\alpha^n, \alpha^{n-1}]$ *is regular and maximal for all* $n \geq 1$. $C_n^{\pm} \in \mathcal{J}$ *are regular and maximal for* $2 \leq n \leq M-2$. *As* $\lim_{n\to} \alpha^n = -\beta$, *regular intervals cover* $[-\beta, \alpha]$. *As the multiplier of* α *is negative, there are no maximal regular intervals of order* 1 *in* $[\alpha, -\alpha]$ ($[\alpha^1, \alpha]$ *and* $-[\alpha^1, \alpha]$ *are the only regular intervals of order* 1).

c) *The image* $P_a(J)$ *of a regular interval* J *of order* $n \geq 1$ *is regular of order* $n-1$ *and* $\widehat{P_a(J)} = P_a(\widehat{J}\,) \subseteq (a, \beta)$. *Conversely, if* J *is* (*maximal*) *regular of order* $n \geq 0$, *and if* $\widehat{J} \subseteq (a, \beta)$, *then both components of* $P_a^{-1}(J)$ *are* (*maximal*) *regular of order* $n+1$.

d) *For each regular interval* J *there are two adjacent regular intervals* J_i, $i = 1, 2$, *with* $\operatorname{int}(J_i) \cap \operatorname{int}(J) = \varnothing$ *and* $ord(J_i) = ord(J) + 3$. *If* $J \in \mathcal{J}$ *there are two adjacent maximal regular intervals* J_i' *with* $\operatorname{int}(J_i') \cap \operatorname{int}(J) = \varnothing$ *and* $ord\, J_i' - ord\, J = \{\pm 1, \pm 3\}$.

Definition 2.3. — For a regular interval J denote by $\mathcal{G}_J : \widehat{A} \to \widehat{J}$ the local inverse to $P_a^{ord(J)}|_J$. Also

$$W_n := \bigcup_{\substack{J \in \mathcal{J} \\ 2 \leq ord(J) \leq n}} \operatorname{int}(J), \quad W = \bigcup_{n \geq 2} W_n$$