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GENERIC POINTS IN THE CARTESIAN POWERS OF THE MORSE DYNAMICAL SYSTEM

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Abstract. — The symbolic dynamical system associated with the Morse sequence is strictly ergodic. We describe some topological and metrical properties of the Cartesian powers of this system, and some of its other self-joinings. Among other things, we show that non generic points appear in the fourth power of the system, but not in lower powers. We exhibit various examples and counterexamples related to the property of weak disjointness of measure preserving dynamical systems.

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Résumé (*Points génériques dans les puissances cartésiennes du système dynamique de Morse*)

Le système dynamique symbolique associé à la suite de Morse est strictement ergodique. Nous décrivons certaines propriétés topologiques et métriques des puissances cartésiennes de ce système, et de certains de ses auto-couplages. Nous montrons en particulier que des points non génériques apparaissent dans la puissance quatrième du système mais n'apparaissent pas dans les puissances inférieures. Nous présentons divers exemples et contre-exemples illustrant la notion de disjonction faible de systèmes dynamiques mesurés.

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Introduction

In this article, we describe ergodic properties of some self-joinings of the dynamical system associated to the Morse sequence. Some of these properties are relevant to the topological dynamics setting, and others are relevant to the measurable dynamics setting.

The dynamical system associated to the Morse sequence, called the *Morse dynamical system* is a well known and widely studied object. We recall its definition and some of its basic properties in Section 1. In this section the classical notions of generic points in dynamical systems and of strict ergodicity are also recalled.

The Morse dynamical system $\mathcal{M}$ is a simple example of a strictly ergodic dynamical system, probably the simplest example after the ergodic translations of compact abelian groups. It has been a surprise for us to discover that non trivial phenomena appear in the genericity properties of points in the Cartesian powers of $\mathcal{M}$: in the Cartesian square and cube of $\mathcal{M}$ every point is generic (for a measure which of course depends on the point) ; but in the fourth Cartesian power of $\mathcal{M}$, this is no longer true. We will see that, in a certain sense, there are a lot of non generic points in the kth Cartesian power of $\mathcal{M}$, when $k \geq 4$.

The study of generic points in Cartesian products of dynamical systems is linked with the notion of *weak-disjointness of dynamical systems*, that has been introduced in [3] and [4] and that we recall now.

DEFINITION. — *Two probability measure preserving dynamical systems* $(X, \mathcal{A}, \mu, T)$ *and* $(Y, \mathcal{B}, \nu, S)$ *are weakly disjoint if, given any function* f *in* $L^2(\mu)$ *and any function* g *in* $L^2(\nu)$, *there exist a set* A *in* $\mathcal{A}$ *and a set* B *in* $\mathcal{B}$ *such that*

- $\mu(A) = \nu(B) = 1$,
- *for all* $x \in A$ *and* $y \in B$, *the following sequence converges:*

$$\Big(\frac{1}{N}\sum_{n=0}^{N-1} f\big(T^n x\big)\cdot g(S^n y)\Big)_{N>0}.$$

This notion of weak disjointness is an invariant of isomorphism in the category of measurable dynamical systems.

If X and Y are compact metric spaces equipped with their Borel σ-algebras and with Borel probabilities μ and ν, it can be shown [4] that the dynamical systems $(X, \mathcal{A}, \mu, T)$ and $(Y, \mathcal{B}, \nu, S)$ are weakly disjoint if and only if the set of points in the Cartesian product $(X \times Y, T \times S)$ that are generic for some measure contains a "rectangle" $A \times B$ of full $\mu \otimes \nu$ measure.

We say that a dynamical system $(X, \mathcal{A}, \mu, T)$ is self-weakly disjoint (of order 2) if it is weakly disjoint from itself. This notion has an obvious k-fold extension, for any integer $k \geq 2$.

From the study of generic points in the Cartesian powers of the Morse dynamical system $\mathcal{M}$, we deduce that this dynamical system is self-weakly disjoint of orders 2 and 3, but not of order ≥ 4. We prove that $\mathcal{M}$ is weakly disjoint from any ergodic k-fold joining of itself (Corollary 2.13), which implies (due to a result of [4]) that *the Morse dynamical system* $\mathcal{M}$ *is weakly disjoint from any ergodic dynamical system*. We do not know if $\mathcal{M}$ is weakly disjoint from every dynamical system.

On the side of "negative" results we prove that the Cartesian square of $\mathcal{M}$ is not self-weakly disjoint (Corollary 3.6) and we prove that most ergodic self-joinings of $\mathcal{M}$ are not self-weakly disjoint (Theorem 3.7). This provides the simplest known example of a non self-weakly disjoint ergodic dynamical system with zero entropy.

We notice here that the Cartesian square of $\mathcal{M}$ is not self-weakly disjoint although it is weakly disjoint from each of its ergodic components (Corollary 2.15).

We give examples of topological dynamical systems in which every point is generic for some measure such that this property fails in the Cartesian square. In the last section of this article we describe the construction of a dynamical system in which every point is generic for some measure and such that this property is preserved in the Cartesian square, but not in the Cartesian cube. We claimed that for the Morse dynamical system this property of genericity is preserved for the cube, but not for the fourth power. We do not know how to construct examples of dynamical systems for which this genericity property is preserved exactly till a given order ≥ 4.

The description of $\mathcal{M}$ as a two point extension of the dyadic odometer plays a major role in our study. The convergence results are deduced from the ergodicity of some cocycles defined on the odometer, the main result here being Proposition 2.10. In order to prove divergence results, we exhibit a particular block structure of a family of four points in the odometer (or equivalently four points in the space $\mathcal{M}$) which give strongly oscillating Birkhoff sums, hence diverging ergodic averages (see Section 3.1).

1. Reminder of some classical notions

1.1. The Morse sequence. — The (Prouhet-Thue-)Morse sequence

$$0\ 1\ 1\ 0\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 0\ 1\ 0\ 1\ \ldots$$

is the sequence $u = (u_n)_{n\geq 0} \in \{0,1\}^{\mathbb{N}}$ defined by one of the following equivalent rules

- $u_0 = 0, u_{2n} = u_n, u_{2n+1} = 1 - u_n$;
- $u_n = 0$ iff there is an even number of 1's in the base 2 expansion of the integer n;
- the sequence begins with a 0 and is a fixed point of the substitution $0 \mapsto 01$, $1 \mapsto 10$;
- $(u_{2^n}, u_{2^n+1}, \ldots, u_{2^{n+1}-1}) = (1-u_0, 1-u_1, \ldots, 1-u_{2^n-1})$ and $u_0 = 0$.

This sequence appears independently in various mathematical works.

In 1851, Prouhet [7] showed that any set of 2^n consecutive integer numbers can be divided into two subsets such that, for any integer k between 0 and $n-1$, the sum of the k-th powers of the elements of one subset is equal to the sum of the k-th powers of the elements of the other subset. He noticed that if we denote by A, resp. B, the set of integers i between 0 and $2^n - 1$ such that

$u_i = 0$, resp $u_i = 1$, then for all k between 0 and $n-1$, for all integers m,

$$\sum_{i\in A}(m+i)^k = \sum_{i\in B}(m+i)^k.$$

At the beginning of the last century, Thue [8] was looking for a sequence "without any cube" and exhibited the sequence u. Indeed, it can be verified that no finite word from u is repeated three times consecutively in u.

Morse [6] introduced the sequence u in his study of recurrence properties of geodesics on some surface with negative curvature. He was interested in the fact that the sequence u is non periodic but minimal: every finite word wich appears once in u appears infinitely often with bounded gaps.

1.2. The Morse dynamical system. — The procedure which associates a dynamical system to the Morse sequence u is standard. We consider the compact space $\{0,1\}^{\mathbb{N}}$ equipped with the shift transformation σ. We denote by K_u the closure of the orbit of the Morse sequence u under σ:

$$K_u := \overline{\{\sigma^n u \,:\, n \in \mathbb{N}\}}.$$

K_u is a compact metrizable space, invariant under σ. The Morse dynamical system is (K_u, σ). The ergodic and spectral properties of this dynamical system have been widely studied by numerous authors including Kakutani, Keane, Kwiatkowski, Lemanczyck. A list of references can be found in [5].

It is known that the dynamical system (K_u, σ) is strictly ergodic (that is minimal and uniquely ergodic). At the level of the sequence u this means that every word that appears once in u appears in u with a strictly positive asymptotic frequency. At the level of the dynamical system (K_u, σ) strict ergodicity means that there exists on K_u a unique σ-invariant probability measure whose support is all of K_u. This measure will be denoted by ν. (A proof of unique ergodicity of the Morse dynamical system is given at the end of this section.)

If $v = (v_n)$ is a sequence of 0's and 1's we denote by $\overline{v}$ the conjugate sequence $\overline{v} = (1 - v_n)$. We remark that K_u is preserved by this conjugacy.

There is a well known and very useful description of the Morse dynamical system as a two point extension of the dyadic odometer. In the more general setting of 'q-multiplicative sequences' this description is explained in, for example, [2]. Let us recall it in the Morse case.

We denote by Ω the topological group of 2-adic integers. It is the set $\{0,1\}^{\mathbb{N}}$ of sequences of 0's and 1's equipped with the product topology making it a compact space and with the additive law of adding with carry (on the right). Formally if $\omega = (\omega_i)_{i\le 0}$ and $\omega' = (\omega'_i)_{i\le 0}$ are two elements of Ω their sum $\omega+\omega'$ is defined by

$$\sum_{i\ge 0}(\omega+\omega')_i 2^i = \sum_{i\ge 0}\omega_i 2^i + \sum_{i\ge 0}\omega'_i 2^i.$$