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THE ACTION SPECTRUM NEAR POSITIVE DEFINITE INVARIANT TORI

by Patrick Bernard

Abstract. — We show that the Birkhoff normal form near a positive definite KAM torus is given by the function α of Mather. This observation is due to Siburg [Si2], [Si1] in dimension 2. It clarifies the link between the Birkhoff invariants and the action spectrum near the torus. Our extension to high dimension is made possible by a simplification of the proof given in [Si2].

Résumé (*Le spectre d'action au voisinage des tores invariants à torsion définie*)
On montre que la forme normale de Birkhoff au voisinage d'un tore KAM à torsion définie est donnée par la fonction α de Mather. Cette observation est due à Siburg [Si2], [Si1], en dimension 2. Elle clarifie le lien entre les coefficients de Birkhoff et le spectre d'action au voisinage du tore. Notre extension à la dimension supérieure est rendue possible par une simplification de la preuve donnée dans [Si2].

1. Introduction

Let us consider a smooth symplectic manifold $(\mathcal{M}, \Omega)$ of dimension $2n$, and a C^1 symplectic diffeomorphism $\phi : \mathcal{M} \to \mathcal{M}$. We are going to study the dynamic in the neighborhood of some invariant tori of ϕ.

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1.1. We note $\mathbb{T}=\mathbb{R}/\mathbb{Z}$. We use the sign $+$ for sums in $\mathbb{R}^n$ or $\mathbb{T}^n$, and we also define $q+v=q+v \bmod \mathbb{Z}^n\in\mathbb{T}^n$ in the standard way when $q\in\mathbb{T}^n$ and $v\in\mathbb{R}^n$. We identify $T^*\mathbb{T}^n$, endowed with its standard symplectic structure, with the product $\mathbb{T}^n\times\mathbb{R}^n$, and use coordinates $(q,p)\in\mathbb{T}^n\times\mathbb{R}^n$. We note $\mathbb{T}^n_0$ the zero section of $T^*\mathbb{T}^n$, that is the submanifold $\mathbb{T}^n\times\{0\}$ of $\mathbb{T}^n\times\mathbb{R}^n$. In the following, we say that a neighborhood of $\mathbb{T}^n_0$ in $\mathbb{T}^n\times\mathbb{R}^n$ is simple if it is fiberwise convex, that is if its intersection with any fiber $q\times\mathbb{R}^n$ is convex. A local chart of an invariant torus $\mathcal{T}$ is a symplectic diffeomorphism from a simple neighborhood of $\mathbb{T}^n_0$ onto its image in $\mathcal{M}$ whose restriction to $\mathbb{T}^n_0$ is a diffeomorphism onto $\mathcal{T}$. Given a torus $\mathcal{T}$, we call simple neighborhood of $\mathcal{T}$ the image of a local chart. There exist local charts of $\mathcal{T}$ if and only if $\mathcal{T}$ is Lagrangian. We identify the vector spaces $H^1(\mathbb{T}^n,\mathbb{R})$ and $H_1(\mathbb{T}^n,\mathbb{R})$ with $\mathbb{R}^n$.

1.2. Definition. — An invariant torus $\mathcal{T}$ is called a C^k *positive definite quasi-periodic invariant torus* if there exists a C^k local chart τ of $\mathcal{T}$, a vector $\omega\in\mathbb{R}^n$ and a positive definite matrix A such that, as $p\to 0$,

$$\tau^{-1}\circ\phi\circ\tau(q,p)=(q+\omega+Ap,p)+o(p).$$

1.3. It follows from the definition above that the torus $\mathcal{T}$ is a Lagrangian invariant torus on which the dynamic is conjugated to a translation. In a simple neighborhood of a quasi-periodic invariant torus $\mathcal{T}$, the diffeomorphism ϕ is homotopic to the identity. It is convenient to choose once and for all a simple neighborhood U_0 of $\mathcal{T}$ and a homotopy ϕ_t between the identity and $\phi_{|U_0}$. We can assume that $\phi_t(\mathcal{T})=\mathcal{T}$ for each t. Let us fix a smaller simple neighborhood U_1 such that $\phi_t(x)\in U_0$ for all $x\in U_1$ and $t\in[0,1]$. A periodic orbit $X=(x_0,x_1,\dots,x_T=x_0)$ of ϕ contained in U_1 has a well defined homology $[X]\in H_1(\mathcal{T},\mathbb{Z})$. This homology is defined by extending the periodic orbit to a periodic curve in U_0 using the homotopy ϕ_t, and by identifying $H_1(U_0,\mathbb{Z})$ with $H_1(\mathcal{T},\mathbb{Z})$.

1.4. Definition. — We say that the invariant torus $\mathcal{T}$ has a Birkhoff normal form of order k if there exists a local chart τ of $\mathcal{T}$ such that

$$\tau^{-1}\circ\phi\circ\tau(q,p)=\big(q+\mathrm{d}h_k(p),p\big)+o_{k-1}(p),$$

where $h_k:\mathbb{R}^n\to\mathbb{R}$ is polynomial of degree k satisfying $h_k(0)=0$. Such a polynomial is called a Birkhoff normal form of degree k near $\mathcal{T}$.

Let us mention the following result which should be seen as a motivation for the definitions above (see [La]).

1.5. Proposition. — *Let ϕ be a smooth symplectomorphism on $\mathcal{M}$, and let $\mathcal{T}$ be a smooth invariant torus. Assume that the dynamic on $\mathcal{T}$ is conjugated to a Diophantine translation, and that $\mathcal{T}$ is Lagrangian* (*this hypothesis is automatic if Ω is exact*). *Then $\mathcal{T}$ admits Birkhoff normal forms to all orders.*

1.6. The restriction of Ω to U_0 is exact. A local Liouville form is a one form λ on U_0 such that $\mathrm{d}\lambda = \Omega$. There exists a local Liouville form λ whose restriction to $\mathcal{T}$ is zero. All the Liouville forms with this property differ by the differential of a function f which is null on $\mathcal{T}$. If λ is such a Liouville form, there exists a unique function g on U_0 such that $\mathrm{d}g = \phi^*\lambda - \lambda$ and $g|_{\mathcal{T}} = 0$. If $\lambda' = \lambda + \mathrm{d}f$ is another Liouville form with the same property, then the associated function is $g' = g + f \circ \phi - f$.

1.7. Let $X = (x_0, x_1, \dots, x_T = x_0)$ be a periodic orbit. We define its action

$$\mathcal{A}(X) = \sum_{i=1}^{T} g(x_i),$$

it is easy to see that this sum does not depend of the choice of the Liouville form λ whose restriction to $\mathcal{T}$ is zero used to define g.

1.8. A periodic orbit X contained in the simple neighborhood U_1, see 1.3, has a period $T(X) \in \mathbb{Z}$, an homology (or rotation number) $[X] \in H_1(\mathcal{T}, \mathbb{Z})$ and an action $\mathcal{A}(X)$. For each simple neighborhood U contained in U_1, let us define the labelled U-action spectrum as the set

$$A_U = \big\{\big(\mathcal{A}(X), T(X), [X]\big)\big\} \subset \mathbb{R} \times \mathbb{Z} \times H_1(\mathcal{T}, \mathbb{Z}),$$

where X ranges over periodic orbits contained in U. For each cohomology class $c \in H^1(\mathcal{T}, \mathbb{R})$, define

$$\alpha_U(c) = \sup_{(a,T,w) \in A_U} \frac{cw - a}{T} \cdot$$

1.9. Theorem. — *Let us consider a C^1 symplectic diffeomorphism ϕ and a C^1 positive definite quasi-periodic invariant torus $\mathcal{T}$. Let us fix a homotpy between ϕ and the identity in a simple neighborhood U_0 of $\mathcal{T}$. For each sufficiently small simple neighborhood U, the function α_U is finite and convex in a neighborhood of 0. The germ at $c = 0 \in H^1(\mathcal{T}, \mathbb{R})$ of the function α_U does not depend on U.*

We call α this common germ. We have defined a germ of function

$$\alpha : H^1(\mathcal{T}, \mathbb{R}) \longrightarrow \mathbb{R},$$

which is a symplectic invariant and depends only on the local labelled action spectrum. Let us call it the averaged energy. This function has been introduced and studied via a slightly different definition by Mather in [Ma], where Theorem 1.9 is already present. We shall present a simple and self contained proof of this result. Note that the finiteness of $\alpha(0)$ is already a non trivial fact which implies the existence of periodic orbits in any neighborhood of $\mathcal{T}$. Our main point here is the relation between the action spectrum, the averaged energy α and the Birkhoff normal forms.

1.10. THEOREM. — *In the setting of Theorem* 1.9, *assume in addition that the torus* $\mathcal{T}$ *has a Birkhoff normal form* h_k *of order* k. *We have*

$$\alpha = h_k \circ \tau^* + o_k$$

where $\tau^* : H^1(\mathcal{T}, \mathbb{R}) \longrightarrow H^1(\mathbb{T}_0^n, \mathbb{R}) = \mathbb{R}^n$ *is the mapping associated to* τ.

1.11. The function α depends on the choice of the homotopy between the identity and ϕ. If another homotopy had been chosen, the associated function $\tilde{\alpha}$ would satisfy

$$\tilde{\alpha}(c) = \alpha(c) + \langle \xi, c \rangle$$

for some $\xi \in H_1(\mathcal{T}, \mathbb{Z})$.

1.12. As a consequence of Theorem 1.10, we obtain a unicity result for Birkhoff normal forms of positive definite tori. Once a homotopy to the identity has been chosen, the normal form is well defined as a polynomial on $H^1(\mathcal{T}, \mathbb{R})$. As a consequence, when seen as a polynomial on $\mathbb{R}^n$, the Birkhoff normal form is well defined only up to the action of $\mathrm{Gl}_n(\mathbb{Z})$.

1.13. It is a straightforward consequence of Theorem 1.10 that the Birkhoff normal form of order k, provided it exists, depends only on the local labelled action spectrum. Recall Proposition 1.5.

1.14. We discuss some sufficient conditions for a quasi-periodic invariant torus to be positive definite in Section 2. We then clarify some conventions concerning the choice of the homotopy ϕ_t and of the rotation vector ω in Section 3. The remaining of the paper is devoted to the proof of the two theorems above.

1.15. We work in coordinates $\mathbb{T}^n \times \mathbb{R}^n$, and first introduce the generating function of our diffeomorphism in Section 4. This generating function is used to define a new action, which is closely connected to the action defined in 1.7 above. In Section 5, we define a new averaged energy using the new action. The links between this new averaged energy and the Birkhoof normal forms are easily established using a property of monotony. This observation is our first improvement compared with the proof of [Si2]. We are then reduced to observe that the two averaged energies coincide. One of the important points is that the germ of the averaged energy is local, *i.e.* depends only of the germ of ϕ along the invariant torus. A similar property was proved in [Si2] using the existence of KAM circles around the elliptic fixed point. It is our second improvement to give a purely variational proof independent of KAM theory. This allows to extend the result to higher dimension where KAM tori do not confine the dynamic. The main step is the existence of a family of periodic orbits near the invariant torus. This property is proved in Section 6. Note that the convexity of the averaged action α is a direct consequence of its definition as a supremum of linear functions.

2. Positive definite KAM tori

The definition of a positive definite quasi-periodic invariant torus given in 1.2 is not intrinsic and requires some comments. Let us consider a smooth symplectic diffeomorphism ϕ and a smooth invariant torus $\mathcal{T}$. We assume that the induced dynamic $\phi_{|\mathcal{T}}$ is conjugated to a Diophantine translation. In addition, we assume that the torus $\mathcal{T}$ is Lagrangian. This hypothesis is automatic if the symplectic form Ω is exact, in view of the following remark of M. Herman (see [He]):

2.1. Let ϕ be a diffeomorphism of an exact symplectic manifold $(\mathcal{M}, \Omega = \mathrm{d}\lambda)$. Let $\mathcal{T}$ be an invariant torus with an ergodic linear induced dynamic. Then the torus $\mathcal{T}$ is Lagrangian. To see this, let us consider an embedding $\eta : \mathbb{T}^n \to \mathcal{M}$ of image $\mathcal{T}$ and such that $R = \eta^{-1}_{|\mathcal{T}} \circ \phi \circ \eta$ is a Diophantine translation. Calling $\mu = \eta^*\lambda$ the restriction of the Liouville form, we have $\mathrm{d}(R^*\mu - \mu) = 0$. Now writing $\mu = a(q)\mathrm{d}q$, we obtain that $a \circ R = a$, this implies that a is constant since R is an ergodic translation. As a consequence, $d\mu = 0$, which means that $\mathcal{T}$ is Lagrangian since $\mathrm{d}\mu = \eta^*\Omega$.

2.2. Since $\mathcal{T}$ is Lagrangian, there exist local charts, this results from a celebrated theorem of Weinstein, see for exemple [DS]. Let τ be a local chart of $\mathcal{T}$ such that $\tau^{-1} \circ \phi_{|\mathcal{T}} \circ \tau_{|\mathbb{T}^n_0}$ is a translation. Then there exist a vector $\omega \in \mathbb{R}^n$ and a smooth function $A : \mathbb{T}^n \longrightarrow \mathcal{S}_n$ of symmetric matrices such that

$$\tau^{-1} \circ \phi \circ \tau(q,p) = \big(q + \omega + A(q)p, p\big) + o(p).$$

In order to see this, note that $\tau^{-1} \circ \phi \circ \tau(q,p) = (q + \omega + A(q)p, B(q)p) + o(p)$, with two function $A, B : \mathbb{T}^n \to M_n$ of matrices. It is then straightforward to check that this mapping is symplectic if and only if $B(q) = \mathrm{Id}$ and $A(q)$ is symmetric for each q.

2.3. It is known that there exists a new chart $\tilde{\tau}$ such that

$$\tilde{\tau}^{-1} \circ \phi \circ \tilde{\tau}(q,p) = (q + \omega + Ap, p) + o(p).$$

where the matrix A is the average

$$A = \int_{\mathbb{T}^n} A(q)$$

with respect to the Haar measure. The chart $\tilde{\tau}$ is obtained by composing τ with an averaging transformation. More precisely, let us consider the time-1 flow ψ of the Hamiltonian $H(q,p) = \frac{1}{2}\langle a(q)p, p\rangle$, where $a(q)$ is a smooth function of symmetric matrices. We have

$$\psi(q,p) = \big(q + a(q)p, p\big) + o(p).$$