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# ON SQUARE FUNCTIONS ASSOCIATED TO SECTORIAL OPERATORS

BY CHRISTIAN LE MERDY

*Dedicated to Alan McIntosh on the occasion of his 60th birthday*

ABSTRACT. — We give new results on square functions

$$\|x\|_F = \Big\| \Big( \int_0^\infty \big|F(tA)x\big|^2 \frac{\mathrm{d}t}{t} \Big)^{1/2} \Big\|_p$$

associated to a sectorial operator $A$ on $L^p$ for $1 < p < \infty$. Under the assumption that $A$ is actually $R$-sectorial, we prove equivalences of the form $K^{-1}\|x\|_G \leq \|x\|_F \leq K\|x\|_G$ for suitable functions $F, G$. We also show that $A$ has a bounded $H^\infty$ functional calculus with respect to $\| \,.\, \|_F$. Then we apply our results to the study of conditions under which we have an estimate $\|(\int_0^\infty |C\mathrm{e}^{-tA}(x)|^2 \mathrm{d}t)^{1/2}\|_q \leq M\|x\|_p$, when $-A$ generates a bounded semigroup $\mathrm{e}^{-tA}$ on $L^p$ and $C\colon D(A) \to L^q$ is a linear mapping.

RÉSUMÉ (*Sur les fonctions carrées associées aux opérateurs sectoriels*)
Nous obtenons de nouveaux résultats sur les fonctions carrées

$$\|x\|_F = \Big\| \Big( \int_0^\infty \big|F(tA)x\big|^2 \frac{\mathrm{d}t}{t} \Big)^{1/2} \Big\|_p$$

associées à un opérateur sectoriel $A$ sur $L^p$ pour $1 < p < \infty$. Quand $A$ est en fait $R$-sectoriel, on montre des équivalences de la forme $K^{-1}\|x\|_G \leq \|x\|_F \leq K\|x\|_G$ pour des fonctions $F, G$ appropriées. On démontre également que $A$ possède un calcul fonctionnel $H^\infty$ borné par rapport à $\| \,.\, \|_F$. Puis nous appliquons nos résultats à l'étude de conditions impliquant une inégalité du type $\|(\int_0^\infty |C\mathrm{e}^{-tA}(x)|^2 \mathrm{d}t)^{1/2}\|_q \leq M\|x\|_p$, où $-A$ engendre un semigroupe borné $\mathrm{e}^{-tA}$ sur $L^p$ et $C\colon D(A) \to L^q$ est une application linéaire.

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## 1. Introduction

The main objects of this paper will be bounded analytic semigroups and sectorial operators on $L^p$-spaces, their $H^\infty$ functional calculus, and their associated square functions. This beautiful and powerful subject grew out of McIntosh's seminal paper [18] and subsequent important works by McIntosh-Yagi [19] and Cowling-Doust-McIntosh-Yagi [6].

We first briefly recall a few classical notions which are the starting point of the whole theory. Given a Banach space $X$, we will denote by $B(X)$ the Banach algebra of all bounded operators on $X$. For any $\omega \in (0,\pi)$, we let

$$\Sigma_\omega = \big\{ z \in \mathbb{C}^* \ ; \ |\mathrm{Arg}(z)| < \omega \big\}$$

be the open sector of angle $2\omega$ around the half-line $(0,\infty)$. Let $A$ be a possibly unbounded operator $A$ on $X$ and assume that $A$ is closed and densely defined. For any $z$ in the resolvent set of $A$ we let $R(z,A) = (z-A)^{-1}$ denote the corresponding resolvent operator. Let $\sigma(A)$ denote the spectrum of $A$. Then by definition, $A$ is *sectorial of type* $\omega$ if the following three conditions are fulfilled:

(S1) $\sigma(A) \subset \overline{\Sigma}_\omega$.

(S2) For any $\theta \in (\omega,\pi)$ there is a constant $K_\theta > 0$ such that

$$\big\| zR(z,A) \big\| \leq K_\theta, \quad z \in \overline{\Sigma}_\theta^c.$$

(S3) $A$ has a dense range.

Very often, (S3) is unnecessary and omitted in the definition of sectoriality. However we include it here to avoid tedious technical discussions. Note the well-known fact that $A$ is one-to-one if it satisfies (S1), (S2) and (S3) above.

Given any $\theta \in (0,\pi)$, we let $H^\infty(\Sigma_\theta)$ be the algebra of all bounded analytic functions $f : \Sigma_\theta \to \mathbb{C}$ and we let $H_0^\infty(\Sigma_\theta)$ be the subalgebra of all $f \in H^\infty(\Sigma_\theta)$ for which there exist two positive numbers $s, c > 0$ such that

$$\big|f(z)\big| \leq c \frac{|z|^s}{(1+|z|)^{2s}}, \quad z \in \Sigma_\theta. \tag{1.1}$$

Now given a sectorial operator $A$ of type $\omega \in (0,\pi)$ on a Banach space $X$, a number $\theta \in (\omega,\pi)$, and a function $f \in H_0^\infty(\Sigma_\theta)$, one may define an operator $f(A) \in B(X)$ as follows. We let $\gamma \in (\omega,\theta)$ be an intermediate angle and consider the oriented contour $\Gamma_\gamma$ defined by

$$\Gamma_\gamma(t) = \begin{cases} -t\mathrm{e}^{i\gamma} & t \in \mathbb{R}_-, \\ t\mathrm{e}^{-i\gamma} & t \in \mathbb{R}_+. \end{cases}$$

Then we let

$$f(A) = \frac{1}{2\pi i} \int_{\Gamma_\gamma} f(z) R(z,A)\,\mathrm{d}z. \tag{1.2}$$

It follows from Cauchy's Theorem that the definition of $f(A)$ does not depend on the choice of $\gamma$ and it can be shown that the mapping $f \mapsto f(A)$ is an algebra homomorphism from $H_0^\infty(\Sigma_\theta)$ into $B(X)$. The next step in $H^\infty$ functional calculus consists in the definition of a possibly unbounded operator $f(A)$ associated to any $f \in H^\infty(\Sigma_\theta)$. Since we shall not use this construction here, we omit it and refer the reader to [18], [19] and [6] for details. We merely recall that by definition, $A$ admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus if $f(A)$ is bounded for any $f \in H^\infty(\Sigma_\theta)$. In that case, the mapping $f \mapsto f(A)$ is a bounded homomorphism from $H^\infty(\Sigma_\theta)$ into $B(X)$, provided that $H^\infty(\Sigma_\theta)$ is equipped with the norm

$$\|f\|_{\infty,\theta} = \sup\bigl\{|f(z)| \; ; \; z \in \Sigma_\theta\bigr\}.$$

We shall be mainly concerned by square functions associated to sectorial operators in the case when $X$ is an $L^p$-space. For any $\omega \in (0, \pi)$, we introduce

$$H_0^\infty(\Sigma_{\omega+}) = \bigcup_{\theta>\omega} H_0^\infty(\Sigma_\theta).$$

Assume first that $X = H$ is a Hilbert space. Given a sectorial operator $A$ of type $\omega$ on $H$ and $F \in H_0^\infty(\Sigma_{\omega+})$, we consider

$$\|x\|_F = \Bigl(\int_0^\infty \|F(tA)x\|^2 \frac{\mathrm{d}t}{t}\Bigr)^{1/2}, \quad x \in H,$$

which may be either finite or infinite. These square function norms were introduced in [18] where it is shown that for any $\theta > \omega$ and any non zero $F \in H_0^\infty(\Sigma_{\omega+})$, $A$ has a bounded $H^\infty(\Sigma_\theta)$ functional calculus if and only if $\|.\|_F$ is equivalent to the original norm of $H$. In [19, Theorem 5], McIntosh-Yagi established the following two remarkable properties. First these square function norms are pairwise equivalent, that is, for any two non zero functions $F$ and $G$ in $H_0^\infty(\Sigma_{\omega+})$ there exists a constant $K > 0$ such that $K^{-1}\|x\|_G \leq \|x\|_F \leq K\|x\|_G$ for any $x \in H$. Second, $A$ always has a bounded $H^\infty$ functional calculus with respect to $\| \,\|_F$. More precisely, for any $\theta > \omega$ and for any $F \in H_0^\infty(\Sigma_\theta)$, there is a constant $K > 0$ such that $\|f(A)x\|_F \leq K\|f\|_{\infty,\theta}\|x\|_F$ for any $f \in H^\infty(\Sigma_\theta)$ and any $x \in H$. Further properties and applications of square functions $\|.\|_F$ were investigated in [3], to which we refer the interested reader.

We now turn to $L^p$-spaces. Let $1 \leq p < \infty$ be a number, let $\Omega$ be an arbitrary measure space, and consider the Banach space $X = L^p(\Omega)$. Given a sectorial operator $A$ of type $\omega$ on $L^p(\Omega)$ and $F \in H_0^\infty(\Sigma_{\omega+})$, we let

$$\|x\|_F = \Bigl\|\Bigl(\int_0^\infty \bigl|F(tA)x\bigr|^2 \frac{\mathrm{d}t}{t}\Bigr)^{1/2}\Bigr\|_{L^p(\Omega)}, \quad x \in L^p(\Omega).$$

Again $\|x\|_F$ may be either finite or infinite. These square function norms were introduced in [6] and play a key role in the study of bounded $H^\infty$ functional calculus on $L^p$-spaces (see Corollary 2.3 below). The latter definition obviously extends the previous one that we recover when $p = 2$. However it is unknown whether the results from [19] reviewed above extend to the case when $p \neq 2$. In particular it is unknown whether square function norms are pairwise equivalent on $L^p$-spaces. In a recent work [2], Auscher-Duong-McIntosh succeded in proving such an equivalence in the case when $-A$ generates a bounded analytic semigroup acting on $L^2(\Omega)$ with suitable upper bounds on its heat kernels. We shall prove that the results from [19, Theorem 5] actually extend to all operators which are not only sectorial but $R$-sectorial. This notion which arose from some recent work of Weis [22] will be explained at the beginning of the next section.

THEOREM 1.1. — *Let $A$ be an $R$-sectorial operator of $R$-type $\omega \in (0,\pi)$ on a space $L^p(\Omega)$, with $1 \leq p < \infty$. Let $\theta \in (\omega,\pi)$ and let $F$ and $G$ be two non zero functions belonging to $H_0^\infty(\Sigma_\theta)$.*

1) *There exists a constant $K > 0$ such that for any $f \in H^\infty(\Sigma_\theta)$ and any $x \in L^p(\Omega)$, we have*

$$\Big\|\Big(\int_0^\infty \big|f(A)F(tA)x\big|^2 \frac{\mathrm{d}t}{t}\Big)^{1/2}\Big\|_{L^p(\Omega)} \leq K\|f\|_{\infty,\theta}\Big\|\Big(\int_0^\infty \big|G(tA)x\big|^2 \frac{\mathrm{d}t}{t}\Big)^{1/2}\Big\|_{L^p(\Omega)}. \tag{1.3}$$

2) *There exists a constant $K > 0$ such that*

$$K^{-1}\|x\|_G \leq \|x\|_F \leq K\|x\|_G, \quad x \in L^p(\Omega).$$

This result will be proved in Section 2 below, where we also include some relevant comments. Then Section 3 is devoted to an application of Theorem 1.1 to the study of $R$-admissibility. This new concept is a natural extension of the classical notion of admissibility considered *e.g.* in [24], [23], [25], [8] or [16]. Given a bounded analytic semigroup $T_t = \mathrm{e}^{-tA}$ on $L^p(\Omega)$ and a linear mapping $C$ from the domain of $A$ into some $L^q(\Sigma)$, we will study conditions under which we have an estimate of the form

$$\Big\|\Big(\int_0^\infty \big|CT_t(x)\big|^2 \mathrm{d}t\Big)^{1/2}\Big\|_{L^q(\Sigma)} \leq M\|x\|_{L^p(\Omega)}.$$

In particular we will show that such an estimate holds if $A$ has a bounded $H^\infty(\Sigma_\theta)$ functional calculus for some $\theta < \frac{1}{2}\pi$ and the set $\big\{(-s)^{1/2}CR(s,A)\ ;\ s \in \mathbb{R},\ s < 0\big\}$ is $R$-bounded. This extends a result of ours ([16]) corresponding to the case when $p = 2$.

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## 2. Equivalence of square function norms

The main purpose of this section is the proof of Theorem 1.1. We first recall the key concepts of $R$-boundedness (see [4]) and $R$-sectoriality (see [22], [21], [14]). Consider a Rademacher sequence $(\varepsilon_k)_{k\geq 1}$ on a probability space $(\Omega_0, \mathbb{P})$. That is, the $\varepsilon_k$'s are pairwise independent random variables on $\Omega_0$ and $\mathbb{P}(\varepsilon_k = 1) = \mathbb{P}(\varepsilon_k = -1) = \frac{1}{2}$ for any $k \geq 1$. Then for any finite family $x_1, \dots, x_n$ in a Banach space $X$, we let

$$\Big\|\sum_{k=1}^{n} \varepsilon_k x_k\Big\|_{\mathrm{Rad}(X)} = \int_{\Omega_0} \Big\|\sum_{k=1}^{n} \varepsilon_k(s) x_k\Big\|_X \mathrm{d}\mathbb{P}(s).$$

Let $X, Y$ be two Banach spaces and let $B(X, Y)$ denote the space of all bounded operators from $X$ into $Y$. By definition, a set $\mathcal{T} \subset B(X, Y)$ is $R$-bounded if there is a constant $C \geq 0$ such that for any finite families $T_1, \dots, T_n$ in $\mathcal{T}$, and $x_1, \dots, x_n$ in $X$, we have

$$\Big\|\sum_{k=1}^{n} \varepsilon_k T_k(x_k)\Big\|_{\mathrm{Rad}(Y)} \leq C \Big\|\sum_{k=1}^{n} \varepsilon_k x_k\Big\|_{\mathrm{Rad}(X)}.$$

In that case, the smallest possible $C$ is called the *$R$-boundedness constant* of $\mathcal{T}$ and is denoted by $R(\mathcal{T})$. If $A$ is a sectorial operator on $X$ and $\omega \in (0, \pi)$ is a number, we say that $A$ is *$R$-sectorial of $R$-type* $\omega$ if for any $\theta \in (\omega, \pi)$, the set $\{zR(z, A)\ ;\ z \in \overline{\Sigma}_\theta^c\} \subset B(X)$ is $R$-bounded.

To describe the range of applications of our result, we first recall that if $X$ is a Hilbert space, then any bounded subset of $B(X)$ is $R$-bounded, hence any sectorial operator of type $\omega$ on $X$ is actually $R$-sectorial of $R$-type $\omega$. Thus Theorem 1.1 comprises [19, Theorem 5] that we recover when $p = 2$. Note that our proof reduces to that of [19] in this case. If $X$ is not isomorphic to a Hilbert space, then there exist bounded subsets of $B(X)$ which are not $R$-bounded (see *e.g.* [1, Proposition 1.13]). The notion of $R$-sectoriality on non Hilbertian Banach spaces is closely related to maximal $L^p$-regularity. Namely, it was proved in [13] and [22] that if $A$ is a sectorial operator of type $< \frac{1}{2}\pi$ on a Banach space $X$ with maximal $L^p$-regularity, then $A$ is $R$-sectorial of $R$-type $< \frac{1}{2}\pi$. Thus the counterexamples to maximal $L^p$-regularity obtained by Kalton-Lancien [13] show that when $p \neq 2$, there exist sectorial operators on $L^p$-spaces which are not $R$-sectorial. Conversely, it was proved in [22] that if $X$ is a UMD Banach space, and $A$ is $R$-sectorial of $R$-type $< \frac{1}{2}\pi$ on $X$, then $A$ has maximal $L^p$-regularity. Thus for $1 < p < \infty$ and $\omega < \frac{1}{2}\pi$, Theorem 1.1 exactly applies when the operator $A$ has maximal $L^p$-regularity. In particular it applies to the operators considered in [2].