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THE STACK OF MICROLOCAL PERVERSE SHEAVES

BY INGO WASCHKIES

ABSTRACT. — In this paper we construct the abelian stack of microlocal perverse sheaves on the projective cotangent bundle of a complex manifold. Following ideas of Andronikof we first consider microlocal perverse sheaves at a point using classical tools from microlocal sheaf theory. Then we will use Kashiwara-Schapira's theory of analytic ind-sheaves to globalize our construction. This presentation allows us to formulate explicitly a global microlocal Riemann-Hilbert correspondence.

RÉSUMÉ (*Le champ des faisceaux pervers microlocaux*). — Nous construisons le champ abélien des faisceaux pervers microlocaux sur le fibré cotangent projectif d'une variété analytique complexe. Suivant des idées d'Andronikof, nous considérons d'abord les germes de faisceaux pervers microlocaux en un point en utilisant les outils classiques de la théorie microlocale des faisceaux. Ensuite nous utilisons la théorie des ind-faisceaux analytiques de Kashiwara-Schapira pour globaliser notre construction. Cette présentation nous permettra de formuler explicitement une version globale de la correspondance de Riemann-Hilbert microlocale.

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INGO WASCHKIES, Université Pierre & Marie Curie (Paris VI), Institut de mathématiques de Jussieu, Équipe analyse algébrique, 175 rue du Chevaleret, 75013 Paris (France)
E-mail : ingo@math.jussieu.fr

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1. Introduction

In [8] Kashiwara constructed the stack of microdifferential modules on a complex contact manifold, generalizing the stack of modules over the ring of microdifferential operators $\mathcal{E}_X$ on the projective cotangent bundle P^*X of a complex manifold X. In his discussion Kashiwara asked for the construction of a stack of microlocal perverse sheaves that should be equivalent to the stack of regular holonomic modules by a microlocal Riemann-Hilbert correspondence. Such a stack should be defined over any field k, but the Riemann-Hilbert morphism only makes sense over $\mathbb{C}$.

There have been several attempts to construct a local version of such a stack. In [1] Andronikof defined a prestack on P^*X and announced the microlocal Riemann-Hilbert correspondence on the stalks. However, at that time there did not exist tools to define a global microlocal Riemann-Hilbert morphism. Another topological construction was proposed in [6], but to our knowledge this project has neither been completed nor published.

Our approach makes use of the theory of analytic ind-sheaves, recently introduced in [12] by Kashiwara and Schapira. Hence, microlocal perverse sheaves on a $\mathbb{C}^{\times}$-conic open subset $U \subset T^*X$ will be ind-sheaves (or more precisely objects of the derived category of ind-sheaves) on U contrary to the construction of [1], in which microlocal perverse sheaves on $U \subset T^*X$ were represented by complexes of sheaves on the base space X. The theory of ind-sheaves provides us with a nice representative of the stack associated to the prestack of [1] and allows us to use the machinery developed in [12]. The essential tool in this description is Kashiwara's functor of ind-microlocalization $\mu : \mathrm{D}^{\mathrm{b}}(k_X) \to \mathrm{D}^{\mathrm{b}}(\mathrm{I}(k_{T^*X}))$ of [9]. This functor enables us to define explicitly a global Riemann-Hilbert morphism when $k = \mathbb{C}$.

In the future, we will hopefully show that we can actually patch (a twisted version of) this stack on a complex contact manifold and prove the Riemann-Hilbert theorem in the complex case.

In more detail, the contents of this paper are as follows.

In Section 2 we recall first the theory of microlocalization of [11] on a real manifold X. We do not review in detail the theory of the micro-support of sheaves but concentrate on the definition of the microlocal category $\mathrm{D}^{\mathrm{b}}(k_X, S)$ where $S \subset T^*X$ is an arbitrary subset. It is defined as the localization of the category $\mathrm{D}^{\mathrm{b}}(k_X)$ by the objects $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}(k_X)$ whose micro-support does not intersect S. For any $\mathcal{F}, \mathcal{G} \in \mathrm{D}^{\mathrm{b}}(k_X)$ we get a natural morphism

$$\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(k_X,S)}(\mathcal{F}, \mathcal{G}) \longrightarrow \mathrm{H}^0\big(S, \mu\mathrm{hom}(\mathcal{F}, \mathcal{G})\big).$$

In the case where $S = \{p\}$, $p \in T^*X$ the category $\mathrm{D}^{\mathrm{b}}(k_X, p)$ has been intensively studied in [11], and in particular it is proved that the morphism above is an isomorphism. We will show that this result is still valid in the category $\mathrm{D}^{\mathrm{b}}(k_X, \{x\} \times \dot{\delta})$ where x is a point of X, $\delta \subset T^*_xX$ a closed cone and $\dot{\delta} = \delta \setminus \{0\}$. Later we will be mainly interested in the case where δ is a complex line. The main tool is the refined microlocal cut-off lemma for non-convex sets, which we recall adding a few comments. We will also need the cut-off functor in Section 5.

Section 3 extends the definitions and results of Section 3 first to $\mathbb{R}$-constructible then to $\mathbb{C}$-constructible sheaves. There are two natural ways to define the microlocalization of the derived category of $\mathbb{R}$-constructible sheaves. We either localize the category $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(k_X)$ by sheaves whose micro-support does not intersect S or we take the full subcategory of $\mathrm{D}^{\mathrm{b}}(k_X, S)$ whose objects are represented by $\mathbb{R}$-constructible sheaves. Following [2] we will use the first definition. One important question is whether or not the two definitions coincide. The main result of this section is that this is the case when $S = \{x\} \times \dot{\delta}$.

In Section 4 we show that the constructions of Section 3 are locally "invariant under quantized contact transformations".

Section 5 is devoted to the study of microlocally $\mathbb{C}$-constructible sheaves in the category $\mathrm{D}^{\mathrm{b}}(k_X, \mathbb{C}^\times p)$. In Section 4 we have shown that the category $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(k_X, \mathbb{C}^\times p)$ is invariant by quantized contact transformation. Hence we are reduced to study microlocally $\mathbb{C}$-constructible sheaves in generic position, *i.e.*, complexes of sheaves whose micro-support is contained in T^*_ZX for a complex (not necessarily smooth) hypersurface Z in a neighborhood of p. We give a complete proof that microlocally $\mathbb{C}$-constructible sheaves in generic position may be represented by $\mathbb{C}$-constructible sheaves (as announced in [1]).

Following [1], we define in Section 6 the category of microlocal perverse sheaves as a full subcategory of $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(k_X, \mathbb{C}^\times p)$. An object $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(k_X, \mathbb{C}^\times p)$ is perverse if for any non-singular point $q \in \mathrm{SS}(\mathcal{F})$ in a neighborhood of $\mathbb{C}^\times p$ the complex $\mathcal{F}$ is isomorphic in $\mathrm{D}^{\mathrm{b}}(k_X, \mathbb{C}^\times q)$ to a constant sheaf $M_Y[d_Y]$ supported on a closed submanifold $Y \subset X$. This definition is natural in view of the microlocal characterization of perverse sheaves of [11] and also leads to the definition of a prestack of microlocal perverse sheaves on P^*X. Then we prove that the category $\mathrm{D}^{\mathrm{b}}_{\mathrm{perv}}(k_X, \mathbb{C}^\times p)$ is abelian as has been announced in [1]. Our proof gives a refined result which allows us to conclude that the stack associated to this prestack is abelian. This stack is the stack of microlocal perverse sheaves on P^*X.

In Section 7 we finally define microlocal perverse sheaves as particular objects of the derived category of ind-sheaves on conic open subsets of T^*X. In Section 6 we have constructed the category of microlocal perverse sheaves at any $p \in P^*X$ (or on $\mathbb{C}^\times p \subset \dot{T}^*X$) which will be equivalent to the stalk of the stack μPerv of microlocal perverse sheaves. The idea of the construction of μPerv is to use the fact Kashiwara's functor μ of ind-microlocalization induces a fully faithful functor from $\mathrm{D}^{\mathrm{b}}_{\mathrm{perv}}(k_X, \mathbb{C}^\times p)$ into the stalk of the prestack of bounded derived categories of ind-sheaves on $\mathbb{C}^\times$-conic subsets of T^*X. Then we can define a microlocal perverse sheaf on a conic open subset $U \subset T^*X$ as an object of $\mathrm{D}^{\mathrm{b}}(\mathrm{I}(k_U))$ that is isomorphic to a microlocal perverse sheaf of $\mathrm{D}^{\mathrm{b}}_{\mathrm{perv}}(k_X, \mathbb{C}^\times p)$ at any point of $p \in U$. We show that the stack of microlocal perverse sheaves is canonically equivalent to the stack associated to the prestack of the last section. Finally, we state without proof the microlocal Riemann-Hilbert theorem which will be the subject of a forthcoming paper.

Appendix A recalls the concepts of 2-limits and 2-colimits in the category of all small categories.

Appendix B gives a short introduction to stacks with emphasis on the special properties resulting from the fact that we work on a topological space. Then we give a criterion for subprestacks of the prestack of derived categories of ind-sheaves on a manifold to be stacks. It is a generalization of a proof of [11] showing that the prestack of perverse sheaves is a substack of the prestack of derived categories of sheaves with $\mathbb{C}$-constructible cohomology. Then we investigate abelian stacks on a topological space. Roughly speaking, an additive stack on a topological space is abelian if and only if its stalks are abelian categories and we have a "lifting property" for kernels and cokernels.

We would like to thank P. Schapira both for having suggested this subject to us, and for always having been ready with precious help, guidance and encouragement throughout the last three years. Secondly, our gratitude goes out to M. Kashiwara with whom we had many invaluable conversations. We would particularly like to thank him for having shared with us his unpublished work on the microlocalization of ind-sheaves. It goes without saying, of course, that we could never have been able to complete this work without either of them. Finally, we would like to thank A. D'Agnolo, P. Polesello, F. Ivorra and D.-C. Cisinski for many useful discussions.

2. Microlocalization of sheaves

2.1. Notations. — Let $\mathbb{R}^+$ denote the group of strictly positive real numbers and $\mathbb{C}^\times$ the group of non-zero complex numbers. We will mainly work on a fixed complex manifold[(1)] X of complex dimension $\dim_{\mathbb{C}} X = d_X$. Let T^*X be

(1) All manifolds (complex or real) in this paper are supposed to be finite dimensional with a countable base of open sets.

its cotangent bundle and $T^*_X X$ the zero section. Set $\dot{T}^*X = T^*X \setminus T^*_X X$ and let $P^*X = \dot{T}^*X/\mathbb{C}^\times$ be the projective cotangent bundle. We denote the natural map by

$$\gamma : \dot{T}^*X \longrightarrow P^*X.$$

If $\Lambda \subset \dot{T}^*X$ is a subset, we define the antipodal set Λ^a as

$$\Lambda^a = \big\{(x;\xi) \mid (x;-\xi) \in \Lambda\big\},$$

and we set

$$\mathbb{R}^+\Lambda = \big\{(x;\xi) \in \dot{T}^*X \mid \exists\, \alpha \in \mathbb{R}^+,\ (x;\alpha\xi) \in \Lambda\big\}.$$

We define similarly $\mathbb{C}^\times\Lambda$. Hence $\mathbb{C}^\times\Lambda = \gamma^{-1}\gamma(\Lambda)$. If $\Lambda = \{p\}$ is a point, we will write $\mathbb{C}^\times p$ instead of $\mathbb{C}^\times\{p\}$.

We say that a subset $\Lambda \subset \dot{T}^*X$ is $\mathbb{R}^+$-conic (resp. $\mathbb{C}^\times$-conic) if it is stable under the action of $\mathbb{R}^+$ (resp. $\mathbb{C}^\times$), *i.e.* if $\mathbb{R}^+\Lambda = \Lambda$ (resp. $\mathbb{C}^\times\Lambda = \Lambda$).

In the sequel, we will often deal with $\mathbb{R}^+$-conic subsets that are only locally $\mathbb{C}^\times$-conic. More precisely, a subset $\Lambda \subset \dot{T}^*X$ is called $\mathbb{C}^\times$-conic at $p \in \dot{T}^*X$ if there exists an open neighborhood U of p such that $U \cap \mathbb{C}^\times\Lambda = U \cap \Lambda$. Note that this definition still makes sense if Λ is a germ of a subset at p. An open subset is always $\mathbb{C}^\times$-conic at each $p \in U$.

Let $S \subset \dot{T}^*X$ be another subset, and suppose that Λ is defined on a germ of a neighborhood of S. Then we say that Λ is $\mathbb{C}^\times$-conic on S if it is $\mathbb{C}^\times$-conic at every point of S. Clearly this is equivalent to the statement that there exists an open neighborhood U of S such that $U \cap \mathbb{C}^\times\Lambda = U \cap \Lambda$. In particular, Λ is $\mathbb{C}^\times$-conic on $\dot{T}^*X$ if and only if it is $\mathbb{C}^\times$-conic.

Finally we call the following easy topological lemma to the reader's attention.

Lemma 2.1.1. — *Let $S \subset \dot{T}^*X$ be a $\mathbb{C}^\times$-conic set and $U \supset S$ an $\mathbb{R}^+$-conic open neighborhood. Then there exists a $\mathbb{C}^\times$-conic open set V such that $S \subset V \subset U$.*

Now let us fix the conventions for sheaves. All sheaves considered here are sheaves of vector spaces over a given field k. We will consider the following categories:

- $\mathrm{D}^{\mathrm{b}}(k_X)$ is the derived category of bounded complexes of sheaves of k vector spaces;
- $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(k_X)$ is the full subcategory of $\mathrm{D}^{\mathrm{b}}(k_X)$ whose objects have $\mathbb{R}$-constructible cohomology;
- $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(k_X)$ is the full subcategory of $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(k_X)$ whose objects have $\mathbb{C}$-constructible cohomology;
- $\mathrm{Perv}(k_X)$ is the full abelian subcategory of $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(k_X)$ whose objects are perverse sheaves. We will follow the conventions for the shift of [11], which imply that a perverse sheaf is concentrated in degrees $-d_X$ to 0.

We will not recall the construction of these categories here, for more details see for instance [11].