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INNER AND OUTER HAMILTONIAN CAPACITIES

by David Hermann

Abstract. — The aim of this paper is to compare two symplectic capacities in $\mathbb{C}^n$ related with periodic orbits of Hamiltonian systems: the Floer-Hofer capacity arising from symplectic homology, and the Viterbo capacity based on generating functions. It is shown here that the inner Floer-Hofer capacity is not larger than the Viterbo capacity and that they are equal for open sets with restricted contact type boundary. As an application, we prove that the Viterbo capacity of any compact Lagrangian submanifold is nonzero.

Résumé (*Capacités hamiltoniennes intérieure et extérieure*). — Nous nous proposons de comparer deux capacités dans $\mathbb{C}^n$ définies par les orbites périodiques de systèmes hamiltoniens. La première est la capacité de Floer-Hofer, issue de l'homologie symplectique ; la seconde est la capacité de Viterbo basée sur des fonctions génératrices. Nous montrons que la capacité intérieure de Floer-Hofer n'est pas plus grande que celle de Viterbo et qu'elles coïncident sur les ouverts dont le bord est une variété de contact restreinte. Nous montrons enfin que la capacité de Viterbo d'une sous-variété lagrangienne compacte n'est jamais nulle.

Contents

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1. Introduction and main results

Throughout this paper, we consider the symplectic space $(\mathbb{C}^n, \omega = \mathrm{d}\lambda_0)$, where $n \geqslant 2$ and $\lambda_0 = \frac{1}{2}\operatorname{Im}(\overline{z} \cdot \mathrm{d}z)$. To each (time-dependent) Hamiltonian function $H \in \mathcal{H}_{\mathrm{t}} = C^\infty(S^1 \times \mathbb{C}^n)$ we associate a Hamiltonian vector field X_H given by $\omega(X_H, .) = -\mathrm{d}H(t, .)$. We shall always assume that X_H is complete: its flow ϕ_t^H is called the Hamiltonian flow of H. We denote the group of compactly supported Hamiltonian diffeomorphisms by $\mathcal{D}$:

$$\mathcal{D} = \big\{\phi_1^H \;/\; H \in C_0^\infty(S^1 \times \mathbb{C}^n)\big\}.$$

The symplectic size of subsets of $\mathbb{C}^n$ is measured by symplectic capacities introduced by Gromov in [6] and developed by Ekeland and Hofer in [3].

Definition 1.1. — A (*relative*) *symplectic capacity* on $(\mathbb{C}^n, \omega)$ is a map which associates a number $c(U) \in [0, +\infty]$ to each subset U of $\mathbb{C}^n$ and which satisfies:

1) $U \subset V \Rightarrow c(U) \leqslant c(V)$ (monotonicity);

2) $c(\phi(U)) = c(U)$ for any $\phi \in \mathcal{D}$ (symplectic invariance);

3) $c(\alpha U) = \alpha^2 c(U)$ for any real number $\alpha > 0$ (homogeneity);

4) $c(B^{2n}(1)) = c(B^2(1) \times \mathbb{C}^{n-1}) = \pi$, where $B^{2n}(1)$ is the unit open ball (normalization).

Given any capacity c, define the associated *inner capacity* $\check{c}$ and *outer capacity* $\hat{c}$ by

$$\text{(1.1)} \qquad \begin{cases} \check{c}(U) = \sup\big\{c(K) \;/\; K \text{ is compact and } K \subset U\big\}, \\ \hat{c}(U) = \inf\big\{c(V) \;/\; V \text{ is open and } \overline{U} \subset V\big\}. \end{cases}$$

The capacity c is said to be *inner regular* if $\check{c} = c$ and *outer regular* if $\hat{c} = c$ (see [8]).

We will consider Hamiltonian capacities in $\mathbb{C}^n$, as introduced in [3]. Given any bounded connected open set $U \subset \mathbb{C}^n$, let $\mathcal{H}_{\mathrm{ad}}(U) \subset \mathcal{H}_{\mathrm{t}}$ be some class of "admissible" Hamiltonian functions. Consider the action functional

$$A_H(\gamma) = \int_{S^1} \gamma^*\lambda_0 - \int_0^1 H\big(t, \gamma(t)\big)\,\mathrm{d}t \quad \text{for } \gamma \in \Lambda = C^\infty(S^1, \mathbb{C}^n)$$

whose critical points are the 1-periodic orbits of X_H. By a universal variational process, select a positive critical value $c(H)$ of A_H for each $H \in \mathcal{H}_{\mathrm{ad}}(U)$. Depending on the functorial properties of $\mathcal{H}_{\mathrm{ad}}$ (see Section 2), define the *capacity* of U by one of the following formulae

$$\begin{cases} c(U) = \sup\big\{c(H) \,/\, H \in \mathcal{H}_{\mathrm{ad}}(U)\big\} & \text{or} \\ c(U) = \inf\big\{c(H) \,/\, H \in \mathcal{H}_{\mathrm{ad}}(U)\big\}. \end{cases} \tag{1.2}$$

Then extend this capacity to all subsets of $\mathbb{C}^n$ by standard processes: the capacity of any open set U is given by

$$c(U) = \sup\big\{c(V) \,/\, V \text{ open, bounded and connected with } V \subset U\big\} \tag{1.3}$$

and the capacity of any subset E in $\mathbb{C}^n$ is given by

$$c(E) = \inf\big\{c(U) \,/\, U \text{ is open and } E \subset U\big\}. \tag{1.4}$$

These capacities have a geometric representation in the following situation.

Definition 1.2. — A hypersurface Σ has *restricted contact type* (or RCT) if there exists a vector field η satisfying $\eta \pitchfork \Sigma$ and $L_\eta \omega = \omega$ on $\mathbb{C}^n$, where L denotes the Lie derivative. A bounded connected open set with RCT boundary will be called a *RCT open set*.

The vector field η is called a *Liouville vector field*. Each Hamiltonian capacity satisfies the Representation Theorem: the capacity of any RCT open set U is the area of some closed characteristic of ∂U.

We will focus here on two of these Hamiltonian capacities. The first one was first defined in [5] using symplectic homology (see [4]). This capacity can be viewed as a variant of the Ekeland-Hofer capacity in [3]. The admissible class $\mathcal{H}_{\mathrm{FH}}(U)$ is the set of those Hamiltonian functions which are negative near $S^1 \times U$ and quadratic at infinity, and the critical value $c_{\mathrm{FH}}(H)$ is obtained by considering the Floer homology groups associated to H. We will also consider the generating function capacity defined by Viterbo in [14]. The admissible class $\mathcal{H}_{\mathrm{V}}(U)$ is the set of compactly supported Hamiltonian functions with support in $S^1 \times U$, and the critical value $c_{\mathrm{V}}(H)$ is defined as a minmax critical value for a generating function of the graph of ϕ_1^H. It should be noticed that the capacity c in [14] is defined *a priori* for disconnected open sets. Thus the capacity c_{V} defined by (1.3) could be smaller than c: if U_1 and U_2 are disjoint open sets, (1.3) shows that $c_{\mathrm{V}}(U_1 \cup U_2) = \max(c_{\mathrm{V}}(U_1), c_{\mathrm{V}}(U_2))$, whereas this property is known for the capacity c only if U_1 and U_2 can be separated by an hyperplane (see [12]). However, this does not affect the results in this paper. A simple observation proves the following regularity result.

Proposition 1.3. — *For any subset U in $\mathbb{C}^n$, we have*

$$c_{\mathrm{V}}(U) = \check{c}_{\mathrm{V}}(U) \quad \text{and} \quad c_{\mathrm{FH}}(U) = \hat{c}_{\mathrm{FH}}(U).$$

Several properties of c_{FH} and c_{V} make the interest of comparing them. Because of Proposition 1.3, it is easy to find open sets U with $c_{\text{FH}}(U) = 1$ and $c_{\text{V}}(U)$ arbitrarily small. But this occurs only because the periodic orbits defining $c_{\text{FH}}(U)$ stay away from U, whereas those defining $c_{\text{V}}(U)$ lie in U. This phenomenon is somehow artificial and it disappears if we compare capacities with the same regularity: our main result is the following inequality.

THEOREM 1.4. — *For any subset U in $\mathbb{C}^n$, we have $\check{c}_{\text{FH}}(U) \leqslant c_{\text{V}}(U)$.*

The main feature in Theorem 1.4 is that $\check{c}_{\text{FH}}$ measures a set from inside, whereas c_{V} measures it from outside, which heuristically explains the inequality. Moreover, by the homogeneity property, any symplectic capacity c satisfies

$$\check{c}(U) = \hat{c}(U) = c(U) \quad \text{for any RCT open set } U \tag{1.5}$$

(see Section 2). This leads to $c_{\text{FH}}(U) \leqslant c_{\text{V}}(U)$, and we will also prove the opposite inequality.

THEOREM 1.5. — *For any RCT open set U in $\mathbb{C}^n$, we have $c_{\text{FH}}(U) = c_{\text{V}}(U)$.*

Our main application of Theorem 1.4 deals with the so-called *Lagrangian camel problem.* Set

$$E_- = \big\{z \in \mathbb{C}^n \,/\, \operatorname{Re}(z_1) < 0\big\}, \quad E_+ = \big\{z \in \mathbb{C}^n \,/\, \operatorname{Re}(z_1) > 0\big\},$$
$$E(\varepsilon) = E_- \cup E_+ \cup B^{2n}(\varepsilon)$$

and consider some compact set $L \subset E_-$. The camel problem is formulated as follows:

Does there exist $H \in \mathcal{H}_t$ with support in $S^1 \times E(\varepsilon)$ satisfying $\phi_1^H(L) \subset E_+$?

By the symplectic reduction properties of the capacity c_V, a positive answer implies $c_V(L) \leqslant \pi\varepsilon^2$ (see [14]). In [10], Théret proved that compact hyperbolic Lagrangian submanifolds and Lagrangian tori have nonzero Viterbo capacity. In other terms, such a submanifold cannot pass through an arbitrarily small hole made in a hyperplane. On the other hand, we can deduce from results by Viterbo in [16] that the Floer-Hofer capacity of any compact Lagrangian submanifold L in $\mathbb{C}^n$ is nonzero. More precisely, let $\mathcal{J}$ be the set of almost complex structures J on $\mathbb{C}^n$ satisfying $J = i$ at infinity and calibrated by ω, which means that $\omega(\,.\,, J.)$ is a Riemannian metric. For each $J \in \mathcal{J}$, consider the set $\mathcal{C}_J$ of J-holomorphic curves with boundary in L. The Gromov Compactness Theorem shows that

$$\widetilde{w}(L) = \sup_{J \in \mathcal{J}} \Big(\inf_{C \in \mathcal{C}_J} \int_C \omega \Big) > 0, \tag{1.6}$$

and the following inequality holds (compare [16], Theorem 6.10).

THEOREM 1.6. — *For any compact Lagrangian submanifold L, we have*

$$c_{FH}(L) \geqslant \widetilde{w}(L).$$

Since we have $\check{c}_{\mathrm{FH}}(L) = c_{\mathrm{FH}}(L)$ for any compact set L (see Section 2), Theorems 1.4 and 1.6 imply the following generalization of [10].

Corollary 1.7. — *If L is a compact Lagrangian submanifold, we have*

$$c_{\mathrm{V}}(L) \geqslant c_{\mathrm{FH}}(L) > 0.$$

Let $[\lambda_0] \in H^1(L)$ be the Liouville class of L and let $\mathcal{P}(L) = [\lambda_0] \cdot \pi_1(L)$ be its periods group: by the Gromov Compactness Theorem, we have $\widetilde{w}(L) \in \mathcal{P}(L)$. When L is rational, that is, $\mathcal{P}(L) = a\mathbb{Z}$ for some real number $a > 0$, Theorem 1.6 implies that L cannot be moved by a Hamiltonian isotopy into an open set with capacity smaller than a, and Theorem 1.7 implies that L cannot pass through a hole of radius less than $\sqrt{a/\pi}$.

In Section 2 we will recall the common features of Hamiltonian capacities, and establish some very elementary results about them, in particular Proposition 1.3 and (1.5). In Section 3 we recall the definitions of symplectic homology and of the Floer-Hofer capacity in [4], [1], [5], [16], [7]. In Section 4 we recall the definition of the Viterbo capacity in [14] and the uniqueness of symplectic homology proved in [15], which is the main tool in the proof of Theorem 1.4. We also give our strategy, explaining how c_{V} and $\check{c}_{\mathrm{FH}}$ can be viewed as differences of critical levels of the same Hamiltonian function. In Section 5 we prove Theorems 1.4, 1.5 and 1.6. The proof of Theorem 1.5 involves the intrinsic description of the capacity c_{FH} given in [16] and [7], followed by a deformation argument. In the proof of Theorem 1.6, we adapt the arguments in [16] to our settings.

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2. Axiomatic properties

Consider some functor $\mathcal{H}_{\mathrm{ad}}$ which associates a class of Hamiltonian functions $\mathcal{H}_{\mathrm{ad}}(U) \subset \mathcal{H}_{\mathrm{t}}$ to each bounded connected open set U in $\mathbb{C}^n$. Set

$$\mathcal{H}_{\mathrm{ad}} = \bigcup_{U \subset \mathbb{C}^n} \mathcal{H}_{\mathrm{ad}}(U),$$

and consider a positive section c of the action spectrum, that is, a map $c : \mathcal{H}_{\mathrm{ad}} \to \mathbb{R}$ such that $c(H) = A_H(\gamma_H)$ is a positive critical value of A_H. Assume that the selector c is invariant by Hamiltonian isotopies, which means that $H \circ \phi \in \mathcal{H}_{\mathrm{ad}}$ and $c(H \circ \phi) = c(H)$ for each $H \in \mathcal{H}_{\mathrm{ad}}$ and each $\phi \in \mathcal{D}$. Given