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RADIATION FIELDS

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Abstract. — We study the "hyperboloidal Cauchy problem" for linear and semi-linear wave equations on Minkowski space-time, with initial data in weighted Sobolev spaces allowing singular behavior at the boundary, or with polyhomogeneous initial data. Specifically, we consider nonlinear symmetric hyperbolic systems of a form which includes scalar fields with a $\lambda\phi^p$ nonlinearity, as well as wave maps, with initial data given on a hyperboloid; several of the results proved apply to general space-times admitting conformal completions at null infinity, as well to a large class of equations with a similar non-linearity structure. We prove existence of solutions with controlled asymptotic behavior, and asymptotic expansions for solutions when the initial data have such expansions. In particular we prove that polyhomogeneous initial data (satisfying compatibility conditions) lead to solutions which are polyhomogeneous at the conformal boundary $\mathcal{I}^+$ of the Minkowski space-time.

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RÉSUMÉ (*Champs rayonnants*). — Nous étudions le « problème de Cauchy hyperboloïdal » pour des équations d'ondes linéaires et semi-linéaires sur l'espace-temps de Minkowski, avec des données initiales, singulières au bord, dans des espaces de Sobolev à poids, où polyhomogènes. Plus précisement, nous considérons une classe de systèmes symétriques hyperboliques non-linéaires, compatibles avec l'équation d'onde scalaire $\lambda\phi^p$, ainsi qu'avec des applications d'onde, avec données initiales prescrites sur un hyperboloide. Plusieurs de nos résultats restent valables pour une classe générale d'espace-temps avec complétions conformes à l'infini isotrope, ainsi que pour une large classe d'équations avec une certaine structure des termes non-linéaires. Nous démontrons l'existence de solutions avec comportement asymptotique contrôlé, ainsi que des développements asymptotiques si les données initiales en possèdent. En particulier nous démontrons, sous une condition de compatibilité, que les données initiales polyhomogènes conduisent à des solutions polyhomogènes près du bord conforme $\mathcal{I}^+$ de l'espace-temps de Minkowski.

Contents

1. Introduction

Bondi *et al.* [6] together with Sachs [34] and Penrose [33], building upon the pioneering work of Trautman [36, 37], have proposed in the sixties a set of boundary conditions appropriate for the gravitational field in the radiation regime. A somewhat simplified way of introducing the Bondi-Penrose (BP) conditions is to assume existence of "asymptotically Minkowskian coordinates" $(x^\mu) = (t, x, y, z)$ in which the space-time metric $\mathfrak{g}$ takes the form

$$\mathfrak{g}_{\mu\nu} - \eta_{\mu\nu} = \frac{\overset{1}{h}_{\mu\nu}(t-r,\theta,\varphi)}{r} + \frac{\overset{2}{h}_{\mu\nu}(t-r,\theta,\varphi)}{r^2} + \cdots, \tag{1.1}$$

where $\eta_{\mu\nu}$ is the Minkowski metric $\mathrm{diag}(-1,1,1,1)$, u stands for $t-r$, with r,θ,φ being the standard spherical coordinates on $\mathbb{R}^3$. The expansion above has to hold at, say, fixed u, with r tending to infinity. Existence of classes of solutions of the vacuum Einstein equations satisfying the asymptotic conditions (1.1) follows from the work in [20] together with [3, 4, 18, 19]. As of today it remains an open problem how general, within the class of radiating solutions of vacuum Einstein equations, are those solutions which display the behavior (1.1). Indeed, the results in [1–4, 17], [17] suggest strongly(1) that a more appropriate setup for such gravitational fields is that of *polyhomogeneous* asymptotic expansions:

$$\mathfrak{g}_{\mu\nu} - \eta_{\mu\nu} \in \mathcal{A}_{\mathrm{phg}}. \tag{1.2}$$

In the context of expansions in terms of a radial coordinate r tending to infinity, the space of polyhomogeneous functions is defined as the set of smooth functions which have an asymptotic expansion of the form

$$f \sim \sum_{i=0}^{\infty}\sum_{j=0}^{N_i} f_{ij}(u,\theta,\varphi)\frac{\ln^j r}{r^{n_i}}, \tag{1.3}$$

for some sequences n_i, N_i, with $n_i \nearrow \infty$. Here the symbol $\sim$ stands for "being asymptotic to": if the right-hand-side is truncated at some finite i, the remainder term falls off appropriately faster. Further, the functions f_{ij} are supposed to be smooth, and the asymptotic expansions should be preserved under differentiation.(2)

The suggestion, that the expansions (1.2) are better suited for describing the gravitational field in the radiation regime than (1.1), arises from the fact that *generic* – in a well defined sense – initial data constructed in [1–4,17], are polyhomogeneous. This leads naturally to the question, whether polyhomogeneity of initial data is preserved under evolution dictated by wave equations.

In this paper we answer in the affirmative this question for semi-linear wave equations, and for the wave map equation, on Minkowski space-time. We develop a functional framework appropriate for the analysis of such questions. We prove local in time existence of solutions for classes of equations that include the semi-linear wave equations and the wave map equation on Minkowski space-time, with conormal and with polyhomogeneous initial data. We show that polyhomogeneity is preserved under evolution when appropriate (necessary) corner conditions are satisfied by the initial data. We note that the initial data considered here are more singular than allowed in the existing related results [7,28,31]. We are planning to analyse the corresponding problems for the vacuum Einstein equation in a forthcoming publication, see also [30].

(1) *Cf.* [29] and references therein for some further related results.

(2) The choice of the sequences n_i, N_i is not arbitrary, and is dictated by the equations at hand. For example, the analysis of $3+1$ dimensional Einstein equations in [17] suggests that consistent expansions can be obtained with $n_i = i$. On the other hand, Theorem 5.7 below gives actually $n_i = \frac{1}{2}i$ for wave-maps on $2+1$ dimensional Minkowski space-time. We note that the $2+1$ dimensional wave map equation is related to the vacuum Einstein equations with cylindrical symmetry (*cf., e.g.*, [5, 14, 15]).

Our main results are the existence and polyhomogeneity of solutions with appropriate polyhomogeneous initial data for the nonlinear scalar wave equation, and for the wave map equation. We achieve this in a few steps. First, we prove local existence of solutions of these equations in weighted Sobolev spaces, *cf.* Theorems 4.1 and 5.1. The next step is to obtain estimates on the time derivatives, *cf.* Theorems 4.4, 5.4 and 5.6. Those estimates are uniform in time in a neighborhood of the initial data surface if the initial data satisfy compatibility conditions. Somewhat surprisingly, we show that all initial data in weighted Sobolev spaces, not necessarily satisfying the compatibility conditions, evolve in such a way that the compatibility conditions will hold on all later time slices; see Corollary 4.5 and Theorems 5.4 and 5.6. Finally, in Theorems 4.10 and 5.7 we prove polyhomogeneity of the solutions with polyhomogeneous initial data; this requires a hierarchy of compatibility conditions. We hope to be able to show in a near future that polyhomogeneity of solutions can be established, for polyhomogeneous initial data, with a finite number of compatibility conditions.

The restriction to Minkowski space-time in Theorem 5.7 is not necessary, and is only made for simplicity of presentation of the results; the same remark applies to Theorem 4.1. Similarly the choice of the initial data hypersurface as the standard unit hyperboloid is not necessary.

This work is organised as follows: First, the reader is referred to Appendix A for definitions, notations, and the functional spaces involved; we also develop calculus in those spaces there. In Section 2 we briefly recall Penrose's conformal completions, as they provide the link between the asymptotic behavior of fields and the local analysis carried on in this work. In Section 3 we consider linear equations. There the key elements of our analysis are: a) Proposition 3.1 and its variations, which give *a priori* estimates in weighted Sobolev spaces; b) the mechanism for proving polyhomogeneity, provided in the proof of Theorem 3.4. The transition from the linear weighted Sobolev estimates to their nonlinear counterparts is done in Sections 4 and 5. This has already been outlined above, and requires a considerable amount of work. In Appendix B we prove several auxiliary results on ODE's, some of which are fairly straightforward; as those results are used in the body of the paper in various, sometimes involved, iterative arguments, it seemed convenient to have precise statements at hand.

Some of the results proved here have been announced in [16].

2. Conformal completions

The aim of this section is to set-up the framework necessary for our considerations; the results here are well known to relativists, but perhaps less so to the PDE community. In any case they are needed to establish notation. Consider, thus, an $n+1$ dimensional space-time $(\mathcal{M}, \mathfrak{g})$ and let

$$\widetilde{\mathfrak{g}} = \Omega^2 \mathfrak{g}. \tag{2.1}$$

Let $\Box_h$ denote the wave operator associated with a Lorentzian metric h,

$$\Box_h f = \frac{1}{\sqrt{|\det h_{\rho\sigma}|}} \partial_\mu \big(\sqrt{|\det h_{\alpha\beta}|}\, h^{\mu\nu} \partial_\nu f\big).$$

We recall that the scalar curvature $R = R(\mathfrak{g})$ of $\mathfrak{g}$ is related to the corresponding scalar curvature $\widetilde{R} = \widetilde{R}(\widetilde{\mathfrak{g}})$ of $\widetilde{\mathfrak{g}}$ by the formula

$$\widetilde{R}\,\Omega^2 = R - 2n \Big\{ \frac{1}{\Omega} \Box_{\mathfrak{g}} \Omega + \frac{n-3}{2} \frac{|\nabla \Omega|^2_{\mathfrak{g}}}{\Omega^2} \Big\}. \tag{2.2}$$

It then follows from (2.2) that we have the identity

$$\Box_{\widetilde{\mathfrak{g}}} \,(\Omega^{-(n-1)/2} f) = \Omega^{-(n+3)/2} \Big(\Box_{\mathfrak{g}} f + \frac{n-1}{4n} (\widetilde{R}\,\Omega^2 - R) f \Big). \tag{2.3}$$

It has been observed by Penrose [33] that the Minkowski space-time $(\mathcal{M}, \eta)$ can be conformally completed to a space-time with boundary $(\widetilde{\mathcal{M}}, \widetilde{\eta})$, $\widetilde{\eta} = \Omega^2 \eta$ on $\mathcal{M}$, by adding to $\mathcal{M}$ two null hypersurfaces, usually denoted by $\mathcal{I}^+$ and $\mathcal{I}^-$, which can be thought of as end points $(\mathcal{I}^+)$ and initial points $(\mathcal{I}^-)$ of inextendible null geodesics [32,33,38]. We will only be interested in "the future null infinity" $\mathcal{I}^+$; an explicit construction (of a subset of $\mathcal{I}^+$) which is convenient for our purposes proceeds as follows: for $(x^0)^2 < \sum_i (x^i)^2$ we define

$$y^\mu = \frac{x^\mu}{x^\alpha x_\alpha} \cdot \tag{2.4}$$

In the coordinate system $\{y^\mu\}$ the Minkowski metric $\eta \equiv -(\mathrm{d}x^0)^2 + (\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2 + (\mathrm{d}x^3)^2 = \eta_{\alpha\beta}\, \mathrm{d}x^\alpha \mathrm{d}x^\beta$ takes the form

$$\eta = \frac{1}{\Omega^2} \eta_{\alpha\beta}\, \mathrm{d}y^\alpha \mathrm{d}y^\beta, \quad \Omega = \eta_{\alpha\beta}\, y^\alpha y^\beta. \tag{2.5}$$

We note that under (2.4) the exterior of the light cone $C_0^{x^\mu} \equiv \{\eta_{\alpha\beta} x^\alpha x^\beta = 0\}$ emanating from the origin of the x^μ-coordinates is mapped to the exterior of the light cone $C_0^{y^\mu} = \{\eta_{\alpha\beta}\, y^\alpha y^\beta = 0\}$ emanating from the origin of the y^μ-coordinates. The conformal completion is obtained by adding $C_0^{y^\mu}$ to $\mathcal{M}$,

$$\widetilde{\mathcal{M}} = \mathcal{M} \cup \big(C_0^{y^\mu} \setminus \{0\} \big),$$

with the obvious differential structure arising from the coordinate system y^μ. We shall use:

- the symbol $\mathcal{I}$ to denote $C_0^{y^\mu} \setminus \{0\}$, and
- $\mathcal{I}^+$ to denote $C_0^{y^\mu} \setminus \{0\} \cap \{y^0 > 0\}$.

As already mentioned, $\mathcal{I}$ so defined is actually a subset of the usual $\mathcal{I}$, but this will be irrelevant for our purposes.