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ESTIMATES OF THE KOBAYASHI-ROYDEN METRIC IN ALMOST COMPLEX MANIFOLDS

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Abstract. — We establish a lower estimate for the Kobayashi-Royden infinitesimal pseudometric on an almost complex manifold (M, J) admitting a bounded strictly plurisubharmonic function. We apply this result to study the boundary behaviour of the metric on a strictly pseudoconvex domain in M and to give a sufficient condition for the complete hyperbolicity of a domain in (M, J).

Résumé (*Estimées de la métrique de Kobayashi-Royden dans les variétés presque complexes*)

Nous établissons une estimée inférieure pour la métrique de Kobayashi-Royden sur une variété presque complexe (M, J) admettant une fonction bornée strictement pluri-sous-harmonique. Nous appliquons ce résultat à l'étude du comportement de la métrique au bord d'un domaine strictement pseudoconvexe dans M et donnons une condition suffisante d'hyperbolicité complète d'un domaine dans (M, J).

Introduction

In the recent paper [11], S. Kobayashi studied the following question: Does every point in an almost complex manifold admit a basis of complete hyperbolic

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neighborhoods? This question was solved in dimension 4 by R. Debalme and S. Ivashkovich in [5].

In the present paper, we give a lower estimate on the Kobayashi-Royden infinitesimal metric on a strictly pseudoconvex domain in an almost complex manifold (such estimates are well-known in the integrable case [7]). A corollary of our main result gives a positive answer to the previous question in any dimension: every point in an almost complex manifold has a complete hyperbolic neighborhood.

Our approach consists of two parts. In order to localize the Kobayashi-Royden metric we use a method developped essentially by N. Sibony [15] in the case of the standard complex structure (see also [6] by K. Diederich-J.E. Fornaess and [10] by N. Kerzman & J.-P. Rosay). This is based on the construction of special classes of plurisubharmonic functions. Then we apply an almost complex analogue of the scaling method due to S. Pinchuk in the integrable case (see, for instance, [14]) and obtain presice estimates of the metric. We point out that similar ideas have been used by F. Berteloot [1], [2] in order to estimate the Kobayashi-Royden metric on some classes of domains in $\mathbb{C}^n$.

We note that S. Ivashkovich and J.-P. Rosay recently proved in [9], among other results, estimates of the Kobayashi-Royden metric similar to ours under weaker assumptions on the regularity of the almost complex structure.

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1. Preliminaries

1.1. Almost complex manifolds. — Let (M', J') and (M, J) be almost complex manifolds and let f be a smooth map from M' to M. We say that f is *(J', J)-holomorphic* if $\mathrm{d}f \circ J' = J \circ \mathrm{d}f$ on TM'. We denote by $\mathcal{O}_{(J',J)}(M', M)$ the set of (J', J)-holomorphic maps from M' to M. Let Δ be the unit disc in $\mathbb{C}$ and J_{st} be the standard integrable structure on $\mathbb{C}^n$ for every n. If $(M', J') = (\Delta, J_{\mathrm{st}})$, we denote by $\mathcal{O}_J(\Delta, M)$ the set $\mathcal{O}_{(J_{\mathrm{st}},J)}(\Delta, M)$ of *J-holomorphic discs* in M.

The following lemma shows that every almost complex manifold (M, J) can be viewed locally as the unit ball in $\mathbb{C}^n$ equipped with a small almost complex deformation of J_{st}. This will be used frequently in the sequel.

LEMMA 1. — *Let (M, J) be an almost complex manifold. Then for every point $p \in M$ and every $\lambda_0 > 0$ there exist a neighborhood U of p and a coordinate diffeomorphism $z : U \to \mathbb{B}$ such that $z(p) = 0$, $\mathrm{d}z(p) \circ J(p) \circ \mathrm{d}z^{-1}(0) = J_{\mathrm{st}}$ and the direct image $z_*(J) := \mathrm{d}z \circ J \circ \mathrm{d}z^{-1}$ satisfies $\|z_*(J) - J_{\mathrm{st}}\|_{\mathcal{C}^2(\overline{\mathbb{B}})} \leq \lambda_0$.*

Proof. — There exists a diffeomorphism z from a neighborhood U' of $p \in M$ onto $\mathbb{B}$ satisfying $z(p) = 0$ and $\mathrm{d}z(p) \circ J(p) \circ \mathrm{d}z^{-1}(0) = J_{\mathrm{st}}$. For $\lambda > 0$ consider the dilation $\mathrm{d}_\lambda : t \mapsto \lambda^{-1} t$ in $\mathbb{C}^n$ and the composition $z_\lambda = \mathrm{d}_\lambda \circ z$. Then $\lim_{\lambda \to 0} \|(z_\lambda)_*(J) - J_{\mathrm{st}}\|_{\mathcal{C}^2(\overline{\mathbb{B}})} = 0$. Setting $U = z_\lambda^{-1}(\mathbb{B})$ for $\lambda > 0$ small enough, we obtain the desired statement. □

1.2. The operators ∂_J and $\bar{\partial}_J$. — Let (M, J) be an almost complex manifold. We denote by TM the real tangent bundle of M and by $T_{\mathbb{C}}M$ its complexification. Recall that $T_{\mathbb{C}}M = T^{(1,0)}M \oplus T^{(0,1)}M$ where

$$T^{(1,0)}M := \{X \in T_{\mathbb{C}}M \,;\; JX = iX\} = \{\zeta - iJ\zeta \,;\; \zeta \in TM\},$$

$$T^{(0,1)}M := \{X \in T_{\mathbb{C}}M \,;\; JX = -iX\} = \{\zeta + iJ\zeta \,;\; \zeta \in TM\}.$$

Let T^*M denote the cotangent bundle of M. Identifying $\mathbb{C} \otimes T^*M$ with $T^*_{\mathbb{C}}M := \mathrm{Hom}(T_{\mathbb{C}}M, \mathbb{C})$ we define the set of complex forms of type $(1,0)$ on M by

$$T_{(1,0)}M = \{w \in T^*_{\mathbb{C}}M \,;\; w(X) = 0, \forall X \in T^{(0,1)}M\}$$

and the set of complex forms of type $(0,1)$ on M by

$$T_{(0,1)}M = \{w \in T^*_{\mathbb{C}}M \,;\; w(X) = 0, \forall X \in T^{(1,0)}M\}.$$

Then $T^*_{\mathbb{C}}M = T_{(1,0)}M \oplus T_{(0,1)}M$. This allows to define the operators ∂_J and $\bar{\partial}_J$ on the space of smooth functions defined on M: given a complex smooth function u on M, we set $\partial_J u = \mathrm{d}u_{(1,0)} \in T_{(1,0)}M$ and $\bar{\partial}_J u = \mathrm{d}u_{(0,1)} \in T_{(0,1)}M$. As usual, differential forms of any bidegree (p, q) on (M, J) are defined by means of the exterior product.

2. J-plurisubharmonic functions with logarithmic singularities

2.1. Plurisubharmonic functions. — We first recall the following definition:

Definition 1. — An upper semicontinuous function u on (M, J) is called J-*plurisubharmonic* on M if the composition $u \circ f$ is subharmonic on Δ for every $f \in \mathcal{O}_J(\Delta, M)$.

If M is a domain in $\mathbb{C}^n$ and $J = J_{\mathrm{st}}$ then a J_{st}-plurisubharmonic function is a plurisubharmonic function in the usual sense.

Definition 2. — Let u be a $\mathcal{C}^2$ function on M, let $p \in M$ and $v \in T_pM$. The *Levi form of u at p*, evaluated on v, is defined by

$$\mathcal{L}^J(u)(p)(v) := -\mathrm{d}(J^\star \mathrm{d}u)(X, JX)(p),$$

where X is any vector field on TM such that $X(p) = v$.

Following [4], [8] we have:

PROPOSITION 1. — *Let u be a $\mathcal{C}^2$ real valued function on M, let $p \in M$ and $v \in T_pM$. Then $\mathcal{L}^J(u)(p)(v) = \Delta(u \circ f)(0)$ where f is any J-holomorphic disc in M satisfying $f(0) = p$, $\mathrm{d}f(0)(\partial/\partial x) = v$.*

Obviously the Levi form is invariant with respect to biholomorphisms. More precisely let u be a $\mathcal{C}^2$ real valued function on M, let $p \in M$ and $v \in T_pM$. If Φ is a diffeomorphism from (M, J) to (M', J'), (J, J')-holomorphic, then

$$\mathcal{L}^J(u)(p)(v) = \mathcal{L}^{J'}(u \circ \Phi^{-1})\big(\Phi(p)\big)\big(\mathrm{d}\Phi(p)(v)\big).$$

Finally it follows from Proposition 1 that a $\mathcal{C}^2$ real valued function u on M is J-plurisubharmonic on M if and only if $\mathcal{L}^J(u)(p)(v) \geq 0$ for every $p \in M$, $v \in T_pM$. This leads to the definition:

DEFINITION 3. — A $\mathcal{C}^2$ real valued function u on M is *strictly J-plurisubharmonic* on M if $\mathcal{L}^J(u)(p)(v)$ is positive for every $p \in M$, $v \in T_pM\backslash\{0\}$.

We have the following example of a J-plurisubharmonic function on an almost complex manifold (M, J):

EXAMPLE 1. — For every point $p \in (M, J)$ there exists a neighborhood U of p and a diffeomorphism $z : U \to \mathbb{B}$ centered at p (*i.e.* $z(p) = 0$) such that the function $|z|^2$ is J-plurisubharmonic on U.

Proof. — Let $p \in M$, U_0 be a neighborhood of p and $z : U_0 \to \mathbb{B}$ be local complex coordinates centered at p, such that $\mathrm{d}z \circ J(p) \circ \mathrm{d}z^{-1} = J_{\mathrm{st}}$ on $\mathbb{B}$. Consider the function $u(q) = |z(q)|^2$ on U_0. For every $w, v \in \mathbb{C}^n$ we have $\mathcal{L}^{J_{\mathrm{st}}}(u \circ z^{-1})(w)(v) = \|v\|^2$. Let $B(0, \frac{1}{2})$ be the ball centered at the origin with radius $\frac{1}{2}$ and let $\mathcal{E}$ be the space of smooth almost complex structures defined in a neighborhood of $\overline{B(0, \frac{1}{2})}$. Since the function $(J', w) \mapsto \mathcal{L}^{J'}(u \circ z^{-1})(w)$ is continuous on $\mathcal{E} \times B(0, \frac{1}{2})$, there exist a neighborhood V of the origin and positive constants λ_0 and c such that $\mathcal{L}^{J'}(u \circ z^{-1})(q)(v) \geq c\|v\|^2$ for every $q \in V$ and for every almost complex structure J' satisfying $\|J' - J_{\mathrm{st}}\|_{\mathcal{C}^2(\bar{V})} \leq \lambda_0$. Let U_1 be a neighborhood of p such that

$$\big\|z_*(J) - J_{\mathrm{st}}\big\|_{\mathcal{C}^2(\overline{z(U_1)})} \leq \lambda_0$$

and let $0 < r < 1$ be such that $B(0, r) \subset V$ and $U := z^{-1}(B(0, r)) \subset U_1$. Then we have the following estimate for every $q \in U$ and $v \in T_qM$:

$$\mathcal{L}^J(u)(q)(v) \geq c\|v\|^2.$$

Then $r^{-1}z$ is the desired diffeomorphism. □

We also have the following

LEMMA 2. — *A function u of class $\mathcal{C}^2$ in a neighborhood of a point p of (M,J) is strictly J-plurisubharmonic if and only there exists a neighborhood U of p with local complex coordinates $z : U \to \mathbb{B}$ centered at p, such that the function $u - c|z|^2$ is J-plurisubharmonic on U for some constant $c > 0$.*

The function $\log|z|$ is J_{st}-plurisubharmonic on $\mathbb{C}^n$ and plays an important role in the pluripotential theory as the Green function for the complex Monge-Ampère operator on the unit ball. In particular, this function is crucially used in Sibony's method in order to localize and estimate the Kobayashi-Royden metric on a complex manifold. Unfortunately, after an arbirarily small general almost complex deformation of the standard structure this function is *not* plurisubharmonic with respect to the new structure (in any neighborhood of the origin), see for instance [4]. So we will need the following statement communicated to the authors by E. Chirka:

LEMMA 3. — *Let p be a point in an almost complex manifold (M,J). There exist a neighborhood U of p in M, a diffeomorphism $z : U \to \mathbb{B}$ centered at p and positive constants λ_0, A, such that $\log|z| + A|z|$ is J'-plurisubharmonic on U for every almost complex structure J' satisfying $\|J' - J\|_{\mathcal{C}^2(\overline{U})} \le \lambda_0$.*

Proof. — Consider the function $u = |z|$ on $\mathbb{B}$. Since

$$\mathcal{L}^{J_{\mathrm{st}}}(u \circ z^{-1})(w)(v) \ge \frac{\|v\|^2}{4|w|}$$

for every $w \in \mathbb{B}\backslash\{0\}$ and every $v \in \mathbb{C}^n$, it follows by a direct expansion of $\mathcal{L}^{J'}(u)$ that there exist a neighborhood U of p, $U \subset\subset U_0$, and a positive constant λ_0 such that $\mathcal{L}^{J'}(u)(q)(v) \ge \|v\|^2/5|z(q)|$ for every $q \in U\backslash\{p\}$, every $v \in T_qM$ and every almost complex structure J' satisfying $\|J' - J\|_{\mathcal{C}^2(\overline{U})} \le \lambda_0$. Moreover, computing the Laplacian of $\log|f|$ where f is any J-holomorphic disc we obtain, decreasing λ_0 if necessary, that there exists a positive constant B such that $\mathcal{L}^{J'}(\log|z|)(q)(v) \ge -B\|v\|^2/|z(q)|$ for every $q \in U\backslash\{p\}$, every $v \in T_qM$ and every almost complex structure J' satisfying $\|J' - J\|_{\mathcal{C}^2(\overline{U})} \le \lambda_0$. We may choose $A = 2B$ to get the result. □

3. Localization of the Kobayashi-Royden metric on almost complex manifolds

Let (M,J) be an almost complex manifold. In what follows we use the notation $\zeta = x + iy \in \mathbb{C}$. According to [13], for every $p \in M$ there is a neighborhood $\mathcal{V}$ of 0 in T_pM such that for every $v \in \mathcal{V}$ there exists $f \in \mathcal{O}_J(\Delta, M)$ satisfying $f(0) = p$, $\mathrm{d}f(0)(\partial/\partial x) = v$. This allows to define the Kobayashi-Royden infinitesimal pseudometric $K_{(M,J)}$.