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KLOOSTERMAN-FOURIER INVERSION FOR SYMMETRIC MATRICES

by Omer Offen

Abstract. — We formulate a Kloosterman transform on the space of generalized Kloosterman integrals on symmetric matrices, and obtain an inversion formula. The formula is a step towards a fundamental lemma of the Jacquet type. At the same time it hints towards a conjectural relative trace formula identity, associated with the metaplectic correspondence.

Résumé (*Inversion de Kloosterman-Fourier pour les matrices symétriques*)
Nous définissons une transformation de Kloosterman sur l'espace des intégrales de Kloosterman généralisées sur les matrices symétriques et nous obtenons une formule d'inversion. Cette formule est une étape vers un lemme fondamental de type de Jacquet. En même temps, elle indique une identité conjecturale de la formule des traces relative associée à la correspondance métaplectique.

1. Introduction

Let F be a non-archimedean local field, $\mathcal{O}_F$ the ring of integers in F and $\wp$ the maximal ideal of $\mathcal{O}_F$. Let $|\cdot|$ denote the normalized absolute value on F so that for a uniformizer ϖ of F we have $|\varpi|^{-1} = \#(\mathcal{O}_F/\wp)$ is the size of the

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residual field. Let ψ be a non-trivial additive character of F. We recall the formula

$$\int \widehat{f}(x)\psi(ax^2)\,\mathrm{d}x = |2a|^{-\frac{1}{2}}\gamma(a,\psi)\int f(x)\psi(-a^{-1}x^2)\,\mathrm{d}x \tag{1}$$

which we use to define the Weil constant γ. Here $a \in F^\times$, $f \in C_c^\infty(F)$ is a Schwartz function on F and $\widehat{f}$ is the Fourier transform of f defined by

$$\widehat{f}(x) = \int f(y)\psi(-2xy)\,\mathrm{d}y.$$

The measure $\mathrm{d}x$ is the self dual Haar measure on F with respect to ψ. Thus, it satisfies

$$f(0) = |2|\int \widehat{f}(x)\,\mathrm{d}x. \tag{2}$$

Let $N = N_n$ be the subgroup of upper triangular unipotent matrices in $\mathrm{GL}_n(F)$. Define the non-degenerate character $\theta = \theta_n$ of N by

$$\theta\Big(u\Big) = \psi\Big(\sum_{i=1}^{n-1} x_{i,i+1}\Big)$$

where $u = (x_{i,j}) \in N$. The Haar measure $\mathrm{d}x$ on F determines a Haar measure on N and a self dual Haar measure on any finite dimensional F-vector space. We will use the measures determined by $\mathrm{d}x$ unless otherwise specified. Denote by $M_{m\times n}(F)$ the set of all $m \times n$ matrices with entries in F. Let

$$M_n(F) = M_{n\times n}(F)$$

and denote by $\mathcal{S} = \mathcal{S}_n$ the space of symmetric matrices

$$\mathcal{S} = \big\{X \in M_n(F);\ {}^tX = X\big\}.$$

We consider the action $(u,s) \mapsto {}^tusu$ of N on $\mathcal{S}$.

Definition 1.1. — An element $s \in \mathcal{S}$ is called *relevant* if θ is trivial on the stabilizer N_s of s in N.

Our objects of interest are the generalized Kloosterman integrals

$$\omega[\Phi,\psi;s] = \int_{N_s\backslash N} \Phi({}^tusu)\theta(u^2)\,\mathrm{d}u \tag{3}$$

for a relevant $s \in \mathcal{S}, \Phi \in C_c^\infty(\mathcal{S})$. Let

$$S_n = \mathcal{S}_n \cap \mathrm{GL}_n(F).$$

The orbits in S_n are fully described in [6]. To describe a set of representatives for all orbits in S_n we view the elements of the Weyl group as permutation matrices in $\mathrm{GL}_n(F)$. Thus a complete set of representatives for the orbits in S_n is the set of all wa, where w is the longest element in the Weyl group of a standard Levi subgroup M of $\mathrm{GL}_n(F)$ and a is in the center of M. All relevant orbits in $\mathcal{S}_n$ with zero determinant contain an element of the form $\left(\begin{smallmatrix} s & \\ & 0\end{smallmatrix}\right)$, where $s \in S_{n-1}$. This is proved in [3] for Hermitian matrices. The proof for symmetric matrices is identical and we omit it here. Representatives for the relevant orbits of zero determinant are therefore given as above in terms of representatives of orbits in S_{n-1}. When $w = 1$ and a is a diagonal matrix, the stabilizer of a in N_n is trivial.

In this sense the diagonal matrices are representatives of the largest orbits. In a sense explained in [3] and [2], the Kloosterman integrals for smaller orbits, *i.e.* with $w \neq 1$ are determined by Kloosterman integrals of the largest orbits. For this reason, our main concern in this work is the space of Kloosterman integrals, restricted to the relevant diagonal matrices. These are of the form $a = \mathrm{diag}(a_1, \ldots, a_n)$ where $a_1, \ldots, a_{n-1} \in F^\times$ and $a_n \in F$. We will denote by $\omega[\Phi, \psi; a_1, \ldots, a_n]$ the Kloosterman integral $\omega[\Phi, \psi; \mathrm{diag}(a_1, \ldots, a_n)]$.

To state our main theorem it will be convenient to introduce a normalization. We introduce the normalizing factors

$$\sigma_n(a) = a_1^{n-1} a_2^{n-2} \cdots a_{n-1},\ \Gamma_n(a, \psi) = \gamma(a_1, \psi)^{n-1} \gamma(a_2, \psi)^{n-2} \cdots \gamma(a_{n-1}, \psi).$$

The normalized Kloosterman integral is

$$\widetilde{\omega}^{\psi}[\Phi, \psi; a_1, \ldots, a_n] = \Gamma_n(-a, \psi) \big|\sigma_n(a)\big|^{\frac{1}{2}} \omega[\Phi, \psi; a_1, \ldots, a_n]. \tag{4}$$

The purpose of the notation $\widetilde{\omega}^{\psi}$ is to emphasize the dependence of the normalization on the character ψ. Let $\Omega_{\psi,n}$ be the space of functions ω on $(F^\times)^{n-1} \times F$ of the form

$$\omega(a_1, \ldots, a_n) = \widetilde{\omega}^{\psi}[\Phi, \psi; a_1, \ldots, a_n]$$

for some $\Phi \in C_c^\infty(\mathcal{S})$. Denote by $[\cdot, \cdot] : F^\times \times F^\times \mapsto \{\pm 1\}$ the quadratic Hilbert symbol. It is defined by the condition $[a, b] = 1$ iff a is representable by the quadratic form $a = x^2 - by^2$. We define the Kloosterman transform $K_{\psi,n}$ on $\Omega_{\psi,n}$ by

$$\begin{aligned} K_{\psi,n}\omega(a_1, \ldots, a_n) = \int \omega(p_1, \ldots, p_n) \psi\Big(- \sum_{i=1}^{n} p_i a_{n+1-i} + \sum_{i=1}^{n-1} \frac{1}{p_i a_{n-i}} \Big) \\ \times \Big(\prod_{i=1}^{n-1} \prod_{j=1}^{n-i} [a_i, p_j] \Big) \mathrm{d}p_n \, \mathrm{d}p_{n-1} \cdots \mathrm{d}p_1 \end{aligned} \tag{5}$$

where the integral over $p_i \in F$, $i = 1, \ldots, n$ is only iterated. Although the integrand is *a priori* only defined for $p_1, \ldots, p_{n-1} \in F^\times$ and $p_n \in F$ we make sense of (5) in the proof of theorem 1.2. Denote by $w_n \in \mathrm{GL}_n(F)$ the permutation matrix with a unit anti-diagonal. For any matrix $X \in M_n(F)$ we will denote by $\mathrm{Tr}(X)$ the trace of X. For a function $\Phi \in C_c^\infty(\mathcal{S})$ let

$$\hat{\Phi}(s) = \int_{\mathcal{S}} \Phi(t) \psi\big(- \mathrm{Tr}(st) \big) \, \mathrm{d}t \tag{6}$$

be the standard Fourier transform of Φ. We will consider the Fourier transform

$$\check{\Phi}(s) = \hat{\Phi}(w_n s w_n) \tag{7}$$

of Φ. Our main theorem is

THEOREM 1.2. — *The integral* (5) *defining the Kloosterman transform on* $\Omega_{\psi,n}$ *is a convergent iterated integral. Moreover, let* $\Phi \in C_c^\infty(\mathcal{S})$ *then,*

$$\begin{aligned}(K_{\psi,n}\widetilde{\omega}^{\psi}[\Phi,\psi;.])(a_1,\dots,a_n)\\ = |2|^{\frac{1}{2}n(n-1)}\gamma(1,\bar{\psi})^{\frac{1}{2}n(n-1)}\widetilde{\omega}^{\bar{\psi}}[\check{\Phi},\bar{\psi};a_1,\dots,a_n].\end{aligned} \tag{8}$$

The theorem shows in particular that $K_{\psi,n}$ is a transform from $\Omega_{\psi,n}$ to $\Omega_{\bar{\psi},n}$. Combining the theorem with Fourier inversion on $\mathcal{S}$ we obtain the inversion of the Kloosterman transform.

COROLLARY 1.3. — *The Kloosterman transform satisfies*

$$K_{\bar{\psi},n} \circ K_{\psi,n} = |2|^{n(n-1)}\,\mathrm{Id}$$

where Id *is the identity map on* $\Omega_{\psi,n}$.

The motivation to the problem lies in a conjectural trace formula identity of the Jacquet type. The identity is concerned with the metaplectic correspondence of [1]. It is a lifting of genuine automorphic representations of the metaplectic double cover $\widetilde{\mathrm{GL}}_n$ of GL_n to automorphic representations of GL_n. Jacquet suggests the following characterization for the image of this lift: A cuspidal automorphic representation of GL_n with trivial central character is a lifting from $\widetilde{\mathrm{GL}}_n$ iff it is (H,χ)-distinguished for some subgroup H of orthogonal similitudes of GL_n and some idéle class quadratic character χ.

For more detail and definitions we refer to [6]. This characterization of the image of metaplectic correspondence will follow from the relative trace formula identity

$$\int K_\Phi(u)\theta(u^2)\,\mathrm{d}u = \int K_f(u_1,u_2)\theta(u_1u_2)\,\mathrm{d}u_1\,\mathrm{d}u_2. \tag{9}$$

Here k is a global field. The integration is over $u, u_1, u_2 \in N_n(k)\backslash N_n(\mathbb{A}_k)$, where on the right hand side N_n is viewed as its splitting in $\widetilde{\mathrm{GL}}_n$, K_f and K_Φ are kernel functions depending on the quadratic character χ, of operators corresponding to the smooth functions of compact support f on $\widetilde{\mathrm{GL}}_n(\mathbb{A}_k)$ and Φ on $S_n(\mathbb{A}_k)$. Again for more details we refer to [6]. The fundamental lemma for this situation is a matching of Kloosterman integrals. If Φ_0 is the characteristic function of $\mathcal{S}\cap K$ where $K = \mathrm{GL}_n(\mathcal{O}_F)$ is the standard maximal compact of $\mathrm{GL}_n(F)$, then $\omega[\Phi_0,\psi;a]$ matches in an appropriate way, an integral of the form

$$\int_{N\times N} \Psi_0({}^t u_1 a u_2)\theta(u_1u_2)\,\mathrm{d}u_1\,\mathrm{d}u_2 \tag{10}$$

where Ψ_0 is the unit element of the genuine spherical Hecke algebra of $\widetilde{\mathrm{GL}}_n(F)$. In [6], Mao proved the fundamental lemma for the case $n = 3$ by brute force computation. In [4], Jacquet developed a method to prove Kloosterman integral identities of the above type. The method requires in both sides an inversion formula for a Fourier-Kloosterman transform on the space of Kloosterman integrals. For the case of a quadratic extension the inversion formulas are obtained in [3] and the method is carried out in [4] to prove the identity of Kloosterman integrals which serves as a fundamental lemma for a relative trace formula. In this work we provide a step towards the fundamental lemma associated with the trace formula (9). At the same time, the formula (5) and mainly the oscillating factor $\prod_{i=1}^{n-1}\prod_{j=1}^{n-i}[a_i, p_j]$ in it, hint to a relation with the metaplectic group. If σ is the 2-cocycle that defines multiplication in $\widetilde{\mathrm{GL}}_n(F)$ as defined in [5], then for $a = \mathrm{diag}(a_1, \ldots, a_n)$ and $p = \mathrm{diag}(p_1, \ldots, p_n)$ we have

$$\sigma(a, w_n p w_n) = \prod_{i=1}^{n-1}\prod_{j=1}^{n-i}[a_i, p_j]. \tag{11}$$

The rest of this manuscript is organized as follows: The main tool we use to prove Theorem 1.2 is Weil's formula. In Chapter 2 we write it in a form convenient for our needs. We then prove the theorem by induction. Chapter 3 provides an inversion formula for some intermediate integrals designed to use an inductive argument. In Chapter 4 the inductive step is carried out to finish the proof of the inversion. Chapter 5 provides a much simpler formula associated with the smallest orbits. We present it here, since once the analogous results for the metaplectic case will be obtained, the method of Jacquet requires this formula in order to prove smooth matching. The proof of the inversion formula closely follows the guidelines of [3], the new ingredient is the occurrence of the Hilbert symbol in the Kloosterman transform. The problem was suggested to me by Jacquet. For the project and for much help and support, I am most thankful to him. Most of this work was written during my visit at IHÉS. I thank the IHÉS for a very pleasant and productive visit.

2. Weil's formula

Let

$$V_{n,m} = \left\{ \begin{pmatrix} 0_n & X \\ {}^tX & 0_m \end{pmatrix};\ X \in M_{n\times m}(F) \right\}.$$

We view $V_{n,m}$ as a self-dual space via the pairing

$$\left(\begin{pmatrix} 0_n & X \\ {}^tX & 0_m \end{pmatrix}, \begin{pmatrix} 0_n & {}^tY \\ Y & 0_m \end{pmatrix} \right) \longmapsto \mathrm{Tr}\left[-\begin{pmatrix} 0_n & X \\ {}^tX & 0_m \end{pmatrix}\begin{pmatrix} 0_n & {}^tY \\ Y & 0_m \end{pmatrix} \right] \tag{12}$$