Bull. Soc. math. France
134 (2), 2006, p. 287–325

INVARIANTS OF REAL SYMPLECTIC FOUR-MANIFOLDS OUT OF REDUCIBLE AND CUSPIDAL CURVES

by Jean-Yves Welschinger

Abstract. — We construct invariants under deformation of real symplectic four-manifolds. These invariants are obtained by counting three different kinds of real rational J-holomorphic curves which realize a given homology class and pass through a given real configuration of (the appropriate number of) points. These curves are cuspidal curves, reducible curves and curves with a prescribed tangent line at some real point of the configuration. They are counted with respect to some sign defined by the parity of their number of isolated real double points and in the case of reducible curves, with respect to some mutiplicity. In the case of the complex projective plane equipped with its standard symplectic form and real structure, these invariants coincide with the ones previously constructed in [11]. This leads to a relation between the count of real rational J-holomorphic curves done in [11] and the count of real rational reducible J-holomorphic curves presented here.

Texte reçu le 5 avril 2005, accepté le 26 octobre 2005.

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2000 Mathematics Subject Classification. — 53D45, 14N35, 14N10, 14P99.

Key words and phrases. — Real symplectic manifold, rational curve, enumerative geometry.

BULLETIN DE LA SOCIÉTÉ MATHÉMATIQUE DE FRANCE 0037-9484/2006/287/$ 5.00

Résumé (*Courbes réductibles, cuspidales et invariants des variétés symplectiques réelles de dimension quatre*)

Nous construisons des invariants par déformation des variétés symplectiques réelles de dimension quatre. Ces invariants sont obtenus en comptant trois différents types de courbes J-holomorphes rationnelles réelles qui réalisent une classe d'homologie donnée et passent par une configuration réelle donnée d'un nombre (adéquat) de points. Ces courbes sont des courbes cuspidales, réductibles et des courbes ayant une tangente prescrite en l'un des points de la configuration. Elles sont comptées en fonction d'un signe qui dépend de la parité du nombre de leurs points doubles réels isolés et, dans le cas des courbes réductibles, en fonction d'une multiplicité. Dans le cas du plan projectif complexe muni de ses formes symplectiques et structures réelles standards, ces invariants coincident avec ceux précédemment construits dans [11]. Ceci mène à une relation entre le comptage de courbes J-holomorphes rationnelles réelles réalisé dans [11] et le comptage de courbes J-holomorphes rationnelles réductibles réelles présenté ici.

Contents

Introduction and statement of the results

Let (X,ω,c_X) be a *real symplectic four-manifold*, that is a triple made of a smooth compact four-manifold X, a symplectic form ω on X and an involution c_X on X such that

$$c_X^*\omega = -\omega.$$

The fixed point set of c_X is called *the real part of* X and is denoted by $\mathbb{R}X$.

A large source of examples is provided by smooth projective surfaces defined by a system of polynomials with real coefficients, the symplectic form is then the restriction of the Fubini-Study form of the ambiant projective space, and the real structure is the restriction of its complex conjugation. Note that the real locus $\mathbb{R}X$ is assumed to be non empty here so that it is a smooth lagrangian surface of (X,ω).

With every such real symplectic four-manifold comes some function

$$\chi : d \in H_2(X;\mathbb{Z}) \longmapsto \chi^d(T) \in \mathbb{Z}[T_1,\dots,T_N],$$

where N denotes the number of connected components of the real locus of the manifold. This function has been constructed in [11] by exctracting integer valued invariants – the coefficients of the polynomial $\chi^d(T)$ – from the following problem of real enumerative geometry: how many real rational curves do realize the homology class d and pass through the adequate number of points?

Remember that for this problem to make sense, we introduce an auxiliary generic almost complex structure J, that is a complex structure on the tangent bundle TX, and we count real rational J-holomorphic curves, that is immersed two-dimensional spheres which are preserved by the involution c_X and whose tangent planes are invariant under J. The adequate number of points is then the expected dimension of this space of real rational J-holomorphic curves, that is $c_1(X)d - 1$, where $c_1(X)$ is the first Chern class of the manifold (X, ω).

Remember that all of these finitely many curves are images of $\mathbb{Z}/2\mathbb{Z}$-equivariant immersions $u : (\mathbb{C}P^1, \text{conj}) \to (X, c_X)$ and the above mentionned invariants are obtained by counting these curves with respect to some sign ± 1 determined by the parity of the number of pairs of complex conjugated points in the set $u^{-1}(\mathbb{R}X)$.

For example, the cubic planar real rational curve parameterized by $t \in \mathbb{C} \mapsto (t^2, t^3 + \epsilon t)$ is counted positively if $\epsilon < 0$ and negatively if $\epsilon > 0$ since $u^{-1}(\mathbb{R}X)$ then contains $\{\pm i\sqrt{\epsilon}\}$, and the pure imaginary planar conic with affine equation $x^2 + y^2 = -1$ is a real rational curve, but not the image of a $\mathbb{Z}/2\mathbb{Z}$-equivariant immersion $u : (\mathbb{C}P^1, \text{conj}) \to (X, c_X)$ since its real part is empty.

Remember finally that if we do not obtain a unique invariant as the Gromov-Witten invariant in the complex case, it is due to the fact the integers we obtain depend on the number of pairs of complex conjugated points in the chosen configuration of $c_1(X)d - 1$ points as well as on the distribution of the remaining points in the different connected components of the real part.

The existence of these invariants raises various questions. Are there analog invariants in higher dimensions? Of which problems of real enumerative geometry is it possible to extract some integer valued invariants? Note that such invariants then bound from below the number of real solutions of the given problem, see Corollary 2.2 of [11]. Does some recursive formula similar to the one obtained by M. Kontsevich for the Gromov-Witten invariants exist?

The works [12] and [10] provide some positive answer to the first question. The present paper, as well as [9] which can be considered as a continuation of this work, is devoted to the study of the next two questions. The problem addressed in [9] is to replace one point condition in the above problem by one tangency condition with some given curve L in the real part $\mathbb{R}X$, as in the classical problem of counting real planar conics tangent to five generic real conics for example. It is proven in [9] that some integer valued invariants can indeed be extracted from this problem, but this requires to take into account other kinds of curves which appear in generic 1-parameter families of curves, namely two components reducible curves, cuspidal curves and curves with some prescribed tangent line at one point of the configuration (or equivalently from Proposition 3.4 of [11], curves having one double point at some point of the configuration). The present paper is actually devoted to the case where L is empty. In this case, only the three terms we have just mentionned occur and indeed they hide some integer valued invariants, see Theorem 0.1. Moreover, these new invariants can be compared with the ones of [11], see Proposition 0.3 below, leading to some relation between the count of generic real rational curves of [11] with the one of real reducible curves done here. However, this relation does not lead to some recursive formula similar to the one obtained in the complex case by M. Kontsevich, see Remark 0.4 below. Note that since the preprint version of this paper and of [9] have appeared, progress has been made on the questions of computation or finding recursion formulas, see [13] and Remark 3 therein.

Let us now come to the precise formulation of the main results of this paper.

We label the connected components of the real part by $(\mathbb{R}X)_1, \dots, (\mathbb{R}X)_N$.

Let $\ell \gg 1$ be an integer large enough and $\mathcal{J}_\omega$ be the space of almost complex structures of X which are tamed by ω and of class C^ℓ. Let $\mathbb{R}\mathcal{J}_\omega$ be the subspace of $\mathcal{J}_\omega$ made of almost complex structures for which the involution c_X is J-anti-holomorphic. These two spaces are separable Banach manifolds which are non empty and contractible (see §1.1 of [11] for the real case).

Let $d \in H_2(X;\mathbb{Z})$ be a homology class satisfying $c_1(X)d > 1$ and set

$$\nu = c_1(X) - 2.$$

Let $\underline{x} = (x_1, \dots, x_\nu) \in X^\nu$ be a *real configuration* of ν distinct points of X, that is an ordered subset of distinct points of X which is globally invariant under c_X. For $j \in \{1, \dots, N\}$, we denote by r_j the number of points in the configuration $\underline{x}$ that are located in the component $(\mathbb{R}X)_j$ and we set

$$r = (r_1, \dots, r_N),$$

so that the N-tuple r encodes the equivariant isotopy class of $\underline{x}$. We will assume throughout the paper that $r \neq (0, \dots, 0)$, see Remark 3.5.

Finally, denote by I the subset of those $i \in \{1, \dots, \nu\}$ for which x_i is fixed by the involution c_X.

For each $i \in I$, choose a line T_i in the tangent plane $T_{x_i}\mathbb{R}X$.

Then, for a generic choice of $J \in \mathbb{R}\mathcal{J}_\omega$, there are only finitely many real rational J-holomorphic curves which realize the homology class d, pass through $\underline{x}$ and are cuspidal. Moreover, these curves are all irreducible and have only transversal double points as well as a unique real ordinary cusp as singularities.

Denote by $\mathcal{C}usp^d(J, \underline{x})$ this finite set of cuspidal curves.

Likewise, there are only finitely many real rational J-holomorphic curves which realize the homology class d, pass through $\underline{x}$ and are reducible. Moreover, these curves have only two irreducible components and only transversal double points as singularities.

Denote by $\mathcal{R}ed^d(J,\underline{x})$ this finite set of reducible curves.

Note that since $I \neq \varnothing$, both irreducible components of such curves are real. Indeed, they would otherwise be exchanged by the involution c_X and would intersect the real locus at only finitely many points. The condition to pass through a point of I would then cost two degrees of liberty instead of one so that generically such curves do not appear. Finally, there are only finitely many real rational J-holomorphic curves which realize the homology class d, pass through $\underline{x}$ and whose tangent line at some point x_i, $i \in I$, is T_i. Moreover, the point x_i having this property is then unique and these curves are all irreducible with only transversal double points as singularities.

Denote by $\mathcal{T}an^d(J,\underline{x})$ this finite set of rational curves.

Note that if $C \in \mathcal{C}usp^d(J,\underline{x}) \cup \mathcal{R}ed^d(J,\underline{x}) \cup \mathcal{T}an^d(J,\underline{x})$, then all the singularities of C are disjoint from $\underline{x}$.

Following [11], we define the *mass* of C and denote by $m(C)$ its number of real isolated double points.

Here, a real double point is said to be *isolated* when it is the local intersection of two complex conjugated branches, whereas it is said to be *non isolated* when it is the local intersection of two real branches.

If C belongs to $\mathcal{R}ed^d(J,\underline{x})$ and C_1, C_2 denote its irreducible components, then we define the *multiplicity* of C, and denote by $\operatorname{mult}(C)$, the number of real intersection points between C_1 and C_2, that is the cardinality of $\mathbb{R}C_1 \cap \mathbb{R}C_2$.

We then set

$$\Gamma_r^d(J,\underline{x}) = \sum_{C \in \mathcal{C}usp^d(J,\underline{x}) \cup \mathcal{T}an^d(J,\underline{x})} (-1)^{m(C)} - \sum_{C \in \mathcal{R}ed^d(J,\underline{x})} (-1)^{m(C)} \operatorname{mult}(C).$$

THEOREM 0.1. — *Let (X,ω,c_X) be a real symplectic four-manifold and*

$$d \in H_2(X;\mathbb{Z}) \quad \textit{be such that} \quad c_1(X)d > 1,\ c_1(X)d \neq 4.$$

The connected components of $\mathbb{R}X$ are labeled by $(\mathbb{R}X)_1,\dots,(\mathbb{R}X)_N$. Let $\underline{x} \subset X$ be a real configuration of $c_1(X)d - 2$ distinct points, r_j be the cardinality of $\underline{x} \cap (\mathbb{R}X)_j$ and $r = (r_1,\dots,r_N)$. Finally, let $J \in \mathbb{R}\mathcal{J}_\omega$ be generic enough so that the integer $\Gamma_r^d(J,\underline{x})$ is well defined. Then, this integer $\Gamma_r^d(J,\underline{x})$ neither depends on the choice of J, nor on the choice of $\underline{x}$.

(The condition $c_1(X)d \neq 4$ is to avoid appearance of multiple curves, see Remark 1.10.)

From this theorem, the integer $\Gamma_r^d(J,\underline{x})$ can be denoted without ambiguity by Γ_r^d, and when it is not well defined, we set $\Gamma_r^d = 0$. We then denote by $\Gamma^d(T)$ the generating function

$$\sum_{r \in \mathbb{N}^N} \Gamma_r^d T^r \in \mathbb{Z}[T_1,\dots,T_N],$$