*Bull. Soc. math. France*
**134** (4), 2006, p. 487–508

# $H^\infty$ CALCULUS AND DILATIONS

BY ANDREAS M. FRÖHLICH & LUTZ WEIS

ABSTRACT. — We characterise the boundedness of the $H^\infty$ calculus of a sectorial operator in terms of dilation theorems. We show e. g. that if $-A$ generates a bounded analytic $C_0$ semigroup $(T_t)$ on a UMD space, then the $H^\infty$ calculus of $A$ is bounded if and only if $(T_t)$ has a dilation to a bounded group on $L^2([0,1],X)$. This generalises a Hilbert space result of C. Le Merdy. If $X$ is an $L^p$ space we can choose another $L^p$ space in place of $L^2([0,1],X)$.

RÉSUMÉ (*Calcul $H^\infty$ et dilatations*). — Nous donnons une condition nécessaire et suffisante en termes de théorèmes de dilatation pour que le calcul $H^\infty$ d'un opérateur sectoriel soit borné. Nous montrons par exemple que, si $A$ engendre un semigroupe $C_0$ analytique borné $(T_t)$ sur un espace UMD, alors le calcul $H^\infty$ de $A$ est borné si et seulement si $(T_t)$ admet une dilatation en un groupe borné sur $L_2([0,1],X)$. Ceci généralise un résultat de C. Le Merdy sur les espaces de Hilbert. Si $X$ est un espace $L_p$, on peut choisir un autre espace $L_p$ à la place de $L_2([0,1],X)$.

## 1. Introduction

In recent years, the holomorphic functional calculus for sectorial operators as introduced in [19] and [4] has received a lot of attention because of its

*Texte reçu le 15 avril 2005, accepté le 25 octobre 2005.*

ANDREAS M. FRÖHLICH, IS ID DC R Analytics, SAP AG, Dietmar-Hopp-Allee 16, 69190 Walldorf, Germany • *E-mail :* a.froehlich@sap.com

LUTZ WEIS, Mathematisches Institut I, Universität Karlsruhe (TH), Englerstraße 2, 76128 Karlsruhe, Germany • *E-mail :* lutz.weis@math.uni-karlsruhe.de

2000 Mathematics Subject Classification. — 47A60, 47A20, 47D06.

Key words and phrases. — $H^\infty$ functional calculus, dilation theorems, spectral operators, square functions, $C_0$ groups, UMD spaces.

applications to evolution equations (e. g., interpolation of domains and maximal regularity [14], [17]) and to Kato's square root problem [1], [2]. In particular, the boundedness of the $H^\infty$ functional calculus was shown for large classes of elliptic differential operators – see [5], [22], and the literature cited there.

One of the first results in this direction was the observation that an accretive operator $A$ on a Hilbert space $H$ has a bounded $H^\infty$ calculus. This follows from the Sz.-Nagy dilation theorem for contractions. More recently, C. Le Merdy [18] has shown that there is a converse to this statement: a sectorial operator $A$ of type $< \frac{1}{2}\pi$ on $H$ has a bounded $H^\infty$ calculus if and only if it is accretive in an equivalent Hilbert space norm, and therefore, by the dilation theorem, $A$ has a bounded $H^\infty$ calculus if and only if there is a second Hilbert space $G$, an isomorphic embedding $J : H \hookrightarrow G$ and a $C_0$ group of isometries $(U_t)$ on $G$ such that

$$JT_t = PU_tJ \quad \text{for all } t > 0, \tag{1.1}$$

where $(T_t)$ is the analytic semigroup generated by $-A$ and $P : G \to J(H)$ is the orthogonal projection onto $J(H)$.

In this paper, we show that this characterisation of the bounded $H^\infty$ calculus can be extended to the class of Banach spaces of finite cotype. These are Banach spaces that do not contain $\ell_n^\infty$ uniformly for all dimensions $n$. If $X$ is a UMD space and $A$ is $R$-sectorial (or almost $R$-sectorial) of type $< \frac{1}{2}\pi$, then our result takes a particular simple form (Corollary 5.4): $A$ has a bounded $H^\infty$ calculus if and only if there is an isomorphic embedding $J : X \to L^2([0,1], X)$, a bounded projection $P : L^2([0,1], X) \to J(X)$ and a group of isometries $(U_t)$ on $L^2([0,1], X)$ such that (1.1) holds. If $X$ is an $L^p(\Omega, \mu)$ space we can even replace $L^2([0,1], X)$ in this statement by another $L^p(\Omega_0, \mu_0)$ space.

Furthermore, our construction shows that the generator of $U_t$ does not just have an $H^\infty$ calculus but can be chosen to be a spectral operator of scalar type in the sense of Dunford and Schwartz [6], and in this form our characterisation also holds in Banach spaces of finite cotype (see Theorem 5.1 and Corollary 5.3). Spectral operators of scalar type are quite rare on Banach spaces that are not Hilbert spaces. Therefore it seems remarkable that the rather large class of operators with a bounded $H^\infty$ calculus can be characterised by dilations to operators in this small class of Banach space operators which have a spectral theory as rich as the spectral theory of normal operators on a Hilbert space.

We also get a dilation theorem for general sectorial operators whose type is not smaller than $\frac{1}{2}\pi$ (Theorem 5.5 and Corollary 5.6). In this case we obtain a result that may be new even in Hilbert space: a sectorial operator on a Hilbert space has a bounded $H^\infty$ calculus if and only if it has a dilation to a normal operator (Corollary 5.7).

Our proofs are based on the square function characterisation of the boundedness of the $H^\infty$ calculus. This technique was introduced for Hilbert spaces by McIntosh [19] and extended to $L^p$ spaces in [4]. (For subspaces of $L^p$ spaces, see also [16].) We use square functions in a general Banach space setting as introduced in [13] and [12]. These definitions and further preliminary information on sectorial operators and spectral theory will be given in Sections 2, 3 and 4. In Section 5 we describe our main results and Section 6 contains the construction of the dilation.

We would like to thank the referee for suggesting several improvements of our presentation.

## 2. $H^\infty$ calculus and spectral operators

We start with some notation. Let $X \neq \{0\}$ be a complex Banach space;

▷ $\mathcal{B}(X)$ will denote the space of all bounded linear operators on $X$ with the operator norm, and

▷ $\mathcal{C}(X)$ is the set of closed linear operators on $X$; we write

▷ $\mathcal{D}(A)$ for the domain of an operator $A$ and $\mathcal{R}(A)$ for its range;

▷ $\mathcal{N}(A)$ is the kernel.

For $\theta \in (0, \pi)$ we define the sector $S_\theta := \{\, z \in \mathbb{C} \setminus \{0\} : |\arg z| < \theta \,\}$. Let

▷ $H(S_\theta)$ be the set of all functions holomorphic on $S_\theta$ and let

▷ $H^\infty(S_\theta)$ be the set of all functions in $H(S_\theta)$ that are bounded.

Furthermore, we define

$$\Psi(S_\theta) := \big\{ \psi \in H(S_\theta) \mid \exists\, c, s > 0,\ \forall z \in S_\theta : |\psi(z)| \leqslant c \min\{|z|^s, |z|^{-s}\} \big\}.$$

Definition 2.1. — An operator $A \in \mathcal{C}(X)$ is *of type* $\mu$, where $\mu \in (0, \pi)$, if

1) $\sigma(A) \subset \overline{S}_\mu$ and

2) for all $\theta \in (\mu, \pi)$, there exists a constant $C_\theta$ such that

$$\big\| R(z, A) \big\| \leqslant C_\theta |z|^{-1} \quad \text{for all } z \notin \overline{S}_\theta .$$

If $A$ is of type $\mu$ for some $\mu \in (0, \pi)$, we say that $A$ is a *sectorial operator*, and by $\omega(A)$ we denote the infimum over all such $\mu$.

Remark 2.2. — An operator $A$ is densely defined and sectorial of type $< \frac{1}{2}\pi$ if and only if $-A$ generates a bounded analytic semigroup [7, Thm 4.6].

Cowling, Doust, McIntosh, and Yagi [4] have introduced a functional calculus for sectorial operators, based on earlier work by McIntosh [19]: if $A$ is of type $\mu$, we can define $\psi(A) \in \mathcal{B}(X)$ for $\psi \in \Psi(S_\theta)$ with $\theta > \mu$ by the contour integral

$$\psi(A) = \frac{1}{2\pi i} \int_{\gamma_\alpha} \psi(z) R(z, A) \mathrm{d}z ,$$

where $\gamma_\alpha$ is the edge of the sector $S_\alpha$ (with $\mu < \alpha < \theta$), oriented in the positive sense. (Compare this with $\psi(\lambda) = \frac{1}{2\pi i}\int_{\gamma_\alpha}\psi(z)(z-\lambda)^{-1}\mathrm{d}z$ for all $\lambda \in S_\alpha$, which follows from Cauchy's formula.)

If, moreover, $A$ has dense domain and dense range (which implies that $A$ is one-to-one, too), this calculus $\Psi(S_\theta) \to \mathcal{B}(X)$ can be extended to the class of functions $f \in H(S_\theta)$ with $\psi_0^n f \in \Psi(S_\theta)$ for some $n \in \mathbb{N}$ and $\psi_0(z) := z/(1+z)^2$ using the definition

$$f(A) := \psi_0(A)^{-n}(\psi_0^n f)(A),$$
$$\mathcal{D}\big(f(A)\big) := \big\{x \in X : (\psi_0 f)(A)y \in \mathcal{R}(\psi_0^n(A))\big\}.$$

Note that $f(A)$ is a densely defined closed operator but not necessarily a bounded operator for all $f \in H^\infty(S_\theta)$.

However, we always have $f_\lambda(A) = R(\lambda, A)$ for $f_\lambda(z) := (\lambda - z)^{-1}$ and $|\arg\lambda| > \mu$. Furthermore,

$$f(A)g(A) = (fg)(A) \qquad \text{on} \quad \mathcal{D}((fg)(A)) \cap \mathcal{D}(g(A)) \text{ and}$$
$$f(A) + g(A) = (f+g)(A) \quad \text{on} \quad \mathcal{D}(f(A)) \cap \mathcal{D}(g(A)).$$

Note that $g(A) = A$ for $g(z) := z$ and $h(tA) = T_t$ for $h(z) := \mathrm{e}^{-z}$ if $-A$ generates a $C_0$ semigroup $(T_t)$ and $\omega(A) < \frac{1}{2}\pi$. We can also define $A^z$ for all $z \in \mathbb{C}$.

Remark 2.3. — This functional calculus has the following convergence property which is an immediate consequence of Lebesgue's convergence theorem: *if $f_n, f \in H^\infty(S_\theta)$ are uniformly bounded and $f_n(z) \to f(z)$ for every $z \in S_\theta$, we have*

$$(f_n\psi)(A) \xrightarrow{n\to\infty} (f\psi)(A) \quad \textit{for every } \psi \in \Psi(S_\theta).$$

This is often used in connection with an "approximate identity" such as

$$\psi_n(z) := \frac{nz - \frac{1}{n}z}{(n+z)(\frac{1}{n}+z)} = -\frac{n}{-n-z} + \frac{\frac{1}{n}}{-\frac{1}{n}-z}$$

which satisfies $\psi_n(A)x \xrightarrow{n\to\infty} x$ for all $x \in \overline{\mathcal{D}(A)} \cap \overline{\mathcal{R}(A)}$.

Definition 2.4. — Let $A$ be of type $\mu$, with dense domain and dense range. We say that $A$ has a *bounded $H^\infty(S_\theta)$ functional calculus*, where $\theta \in (\mu, \pi)$, if $f(A) \in \mathcal{B}(X)$ for all $f \in H^\infty(S_\theta)$. By $\omega_{H^\infty}(A)$ we denote the infimum over all such $\theta$.

In this case, there exists a constant $C$ such that $\|f(A)\| \leqslant C\|f\|_\infty$ for all $f \in H^\infty(S_\theta)$. To check that $A$ has a bounded $H^\infty(S_\theta)$ functional calculus it suffices [4, Cor. 2.2] to show $\|\psi(A)\| \leqslant C\|\psi\|_\infty$ for all $\psi \in \Psi(S_\theta)$.

If a sectorial operator has a bounded $H^\infty$ functional calculus (short: a bounded $H^\infty$ calculus), we always have weak estimates of the following form:

Proposition 2.5 (see [4, Thm 4.2]). — *Let $A$ be a sectorial operator of type $\mu$ with dense domain and dense range. If $A$ has a bounded $H^\infty(S_\theta)$ functional calculus for some $\theta \in (\mu, \pi)$, then for every $\psi \in \Psi(S_\theta)$, there exists a constant $C > 0$ satisfying*

$$\int_0^\infty \big|\big\langle \psi(tA)x, x'\big\rangle\big| \frac{\mathrm{d}t}{t} \leqslant C\|x\| \cdot \|x'\| \quad \textit{for all } x \in X \textit{ and } x' \in X'.$$

For some sectorial operators we can define $f(A) \in \mathcal{B}(X)$ for every bounded Borel function $f \in B_b(\sigma(A))$ on $\sigma(A)$ and get a functional calculus which has the same properties as the functional calculus of normal operators on Hilbert spaces.

Definition 2.6. — A sectorial operator $A$ is said to be a *spectral operator of scalar type* if there exists a functional calculus $\Phi : B_b(\sigma(A)) \to \mathcal{B}(X)$ with the following properties:

1) The operator $\Phi$ is bounded, linear and multiplicative.
2) For $\lambda \notin \sigma(A)$ we have $\Phi((\lambda - \cdot)^{-1}) = R(\lambda, A)$, and $\Phi(1) = \mathrm{id}_X$.
3) Let $f_n, f \in B_b(\sigma(A))$ and $f_n(z) \to f(z)$ for almost all $z \in \sigma(A)$. If $\|f_n\|_\infty \leqslant C$ for all $n \in \mathbb{N}$, then $\Phi(f_n)x \to \Phi(f)x$ for all $x \in X$.

Such operators are studied in detail in [6, Section XVIII.2.8], where a definition in terms of spectral representations and spectral measures is given. However, Theorem 11 in [6, XVIII.2.8], shows that our definition is equivalent. Of course, the strong functional calculus of Definition 2.6 implies the boundedness of the $H^\infty$ calculus.

Proposition 2.7. — *If $A$ is a spectral operator of scalar type with $\sigma(A) \subset \overline{S}_\mu$, then $A$ has a bounded $H^\infty(S_\theta)$ functional calculus for every $\theta > \mu$.*

*Proof.* — Obviously, $A$ is sectorial of type $\mu$. We will show that $\Phi(\psi) = \psi(A)$ for all $\psi \in \Psi(S_\theta)$, where $\psi(A)$ is the operator defined via the functional calculus for sectorial operators. (The boundedness of $\Phi$ gives the desired result then.) By property 2) we have

$$\psi(A) = \frac{1}{2\pi i}\int_\gamma \psi(z)R(z, A)\mathrm{d}z = \frac{1}{2\pi i}\int_\gamma \psi(z)\Phi\big((z - \cdot)^{-1}\big)\mathrm{d}z,$$

and using linearity of $\Phi$ and the convergence property 3) the claim follows. (The latter can be applied because the existence of the integrals is clear.) □

## 3. The moon dual

Let $A \in \mathcal{C}(X)$ be densely defined. Thus, if $A$ is sectorial, the dual operator $A' \in \mathcal{C}(X')$ is sectorial, too, but in general, it will not be densely defined. Consequently, we can not plug $A'$ into the functional calculus.